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JÓZEF DUDA

Funkcjonały Lapunowa dla układów z opóźnieniem

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The Lyapunov functionals for time delay systems

Summary

In this monograph are presented results of the author's research on the determination of the Lyapunov functionals for linear systems with time delay and its applications in the parametric optimization problem. The Lyapunov quadratic functionals are used to calculation of a value of a quadratic performance index of quality in the process of parametric optimization for time delay systems. The value of that functional at the initial state of the time delay system is equal to the value of a quadratic performance index of quality. To calculate the value of a performance index of quality one needs the formulas for the Lyapunov functional coefficients. In this monograph the method proposed by Repin [79] is applied to obtain the Lyapunov functionals, with coefficients given by analytical formulas. In Chapter 2. are considered systems with the retarded type time delay. This method is applied to the system with one delay (Chapter 2.2), to the system with two delays (Chapter 2.3), to the retarded type time delay system with both lumped and distributed delay (Chapter 2.4), to the system with a retarded type time-varying delay (Chapter 2.5). In Chapter 3. are considered neutral systems. Repin's method is applied to the neutral system with lumped delay (Chapter 3.2), to the neutral system with both lumped and distributed delay (Chapter 3.3) and to the neutral system with a time-varying delay (Chapter 3.4). In last years a method of determination of a Lyapunov functional by means of Lyapunov matrices is very popular, see for example [50–66, 72, 73, 76, 81–83]. This method is applied to the parametric optimization problem of retarded type time delay system both with one and two delays (Chapter 4) and to the parametric optimization problem of neutral type time delay system for system with one and two delays (Chapter 5). The examples of using of the Lyapunov functionals to calculation of the performance index value in the parametric optimization problem for linear systems with time delay are given.

JÓZEF DUDA

Funkcjonały Lapunowa dla układów z opóźnieniem

Streszczenie

W monografii przedstawiono wyniki badań autora nad określeniem funkcjonałów Lapunowa dla liniowych układów z opóźnieniem i ich zastosowaniem w procesie optymalizacji parametrycznej. Kwadratowe funkcjonały Lapunowa są stosowane do wyznaczenia wartości kwadratowego wskaźnika jakości w procesie optymalizacji parametrycznej układów z opóźnieniem. Wartość funkcjonału dla stanu początkowego układu z opóźnieniem jest równa wartości kwadratowego wskaźnika jakości. Do wyznaczenia wartości wskaźnika jakości konieczna jest znajomość wzorów na współczynniki funkcjonału Lapunowa. W monografii została zastosowana metoda, zaproponowana przez Repina [79], wyznaczenia wzorów na współczynniki funkcjonału Lapunowa. W rozdziale 2. dla układu z opóźnieniem. W kolejnych podrozdziałach została zastosowana metoda Repina do wyznaczania współczynników funkcjonału Lapunowa dla układu z jednym opóźnieniem skupionym (rozdział 2.2), dla układu z dwoma skupionymi opóźnieniami (rozdział 2.3), dla układu z opóźnieniem skupionym i rozłożonym (rozdział 2.4), dla układu z opóźnieniem zmiennym w czasie (rozdział 2.5). W rozdziale 3. zastosowano metodę Repina dla układu neutralnego. Kolejno dla układu neutralnego z opóźnieniem skupionym (rozdział 3.2), dla układu neutralnego z opóźnieniem skupionym i rozłożonym (rozdział 3.3) oraz dla układu neutralnego z opóźnieniem zmiennym w czasie (rozdział 3.4). W ostatnich latach jest bardzo popularna metoda wyznaczania funkcjonału Lapunowa za pomocą macierzy Lapunowa, patrz np. [50–66, 72, 73, 76, 81–83]. Ta metoda została zastosowana w procesie optymalizacji parametrycznej dla układu z jednym i dwoma opóźnieniami (rozdział 4) i w procesie optymalizacji parametrycznej dla układu neutralnego z jednym i z dwoma opóźnieniami (rozdział 5). W monografii zostały również przedstawione przykłady zastosowania funkcjonałów Lapunowa do obliczania wartości wskaźnika jakości w procesie optymalizacji parametrycznej układów z opóźnieniem.

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Notations and symbols

$\mathbb{R}$ – is the set of all real numbers
$\mathbb{C}$ – is the set of all complex numbers
$\mathbb{R}^n$ – is a space of all n-vectors with entries n $\mathbb{R}$
$\mathbb{R}^{n\times m}$ – is a space of all $n\times m$ real-valued matrices
$I, I_{n\times n}$ – is an identity matrix, identity $n\times n$ matrix
$0_{n\times m}$ – is a zero $n\times m$ matrix
0_r – is the $\mathbb{R}^n$-valued trivial function, $0_r(\theta)=0\in\mathbb{R}^n,\ \theta\in[-r,0]$
A^T – transpose of a matrix A
$A>0$ – symmetric matrix A is positive definite
$A\otimes B$ – is a Kronecker product of matrices A and B
$col\,A$ – is a column vector which consists of columns of matrix A
$\lambda(C)$ – is the eigenvalue of the matrix C
$\sigma(C)$ – is a spectrum of matrix C and is defined as $\sigma(C)=\{\lambda\in\mathbb{C}:\det(\lambda I-C)=0\}$
$\gamma(C)$ – is the spectral radius of a matrix C and is defined as $\gamma(C)=\sup\{|\lambda|:\lambda\in\sigma(C)\}$
$\|\cdot\|_{\mathbb{R}^n}$ – is an Euclidean norm in $\mathbb{R}^n$
$C([-r,0],\mathbb{R}^n)$ – is a space of all continuous $\mathbb{R}^n$ valued functions defined on the segment $[-r,0]$ with the uniform norm $\|\varphi\|_C=\sup\limits_{\theta\in[-r,0]}\|\varphi(\theta)\|$
$C^1([-r,0],\mathbb{R}^n)$ – is a space of all continuous $\mathbb{R}^n$ valued functions with continuous derivative defined on the segment $[-r,0]$
$L^2([-r,0),\mathbb{R}^n)$ – is a space of all Lebesgue square integrable functions defined on the segment $[-r,0)$ with values in $\mathbb{R}^n$
$\|\varphi\|_{L^2}$ – is a norm in $L^2([-r,0),\mathbb{R}^n)$; ($\|\varphi\|_{L^2}=\sqrt{\int_{-r}^{0}\left(\|\varphi(t)\|^2_{\mathbb{R}^n}\right)dt}$)
$W^{1,2}([-r,0),\mathbb{R}^n)$ – is a space of all absolutely continuous functions with derivatives in a space of Lebesgue square integrable functions on interval $[-r,0)$ with values in $\mathbb{R}^n$

$\| \varphi \|_{W^{1,2}}$ – is a norm in $W^{1,2}([-r,0),\mathbb{R}^n)$;
($\| \varphi \|_{W^{1,2}} = \sqrt{\int_{-r}^{0} \left(\| \varphi(t) \|_{\mathbb{R}^n}^2 + \| \frac{d\varphi(t)}{dt} \|_{\mathbb{R}^n}^2 \right) dt}$)

$PC([-r,0],\mathbb{R}^n)$ – is a space of all piece-wise continuous vector valued functions defined on the segment $[-r,0]$ with the uniform norm $\| \varphi \|_{PC} = \sup_{\theta \in [-r,0]} \| \varphi(\theta) \|$

$PC^1([-r,0],\mathbb{R}^n)$ – is a space of all piece-wise continuously differentiable vector valued functions defined on the segment $[-r,0]$ with the uniform norm $\| \varphi \|_{PC^1} = \sup_{\theta \in [-r,0]} \| \varphi(\theta) \|$

$x_t(t_0,\varphi) : [-h,0] \to \mathbb{R}^n$ – is a shifted restriction of the function $x(\cdot,t_0,\varphi)$ to an interval $[t-h,t]$ and is defined by a formula $x_t(t_0,\varphi)(\theta) := x(t+\theta,t_0,\varphi)$ for $t \geq t_0$ and $\theta \in [-h,0]$

$x_t(\varphi) : [-h,0] \to \mathbb{R}^n$ – is a shifted restriction of the function $x(\cdot,\varphi)$ to an interval $[t-h,t]$ and is defined by a formula $x_t(\varphi)(\theta) := x(t+\theta,\varphi)$ for $t \geq 0$ and $\theta \in [-h,0]$

$x_t : [-h,0] \to \mathbb{R}^n$ – is a shifted restriction of the function $x(\cdot,\varphi)$ to an interval $[t-h,t]$ and is defined by a formula $x_t(\theta) := x(t+\theta)$ for $t \geq 0$ and $\theta \in [-h,0]$, when the function φ is known

$f(t+0)$ – is the right-hand-side limit of $f(t)$ at a point t,
$f(t+0) = \lim_{\varepsilon \to 0} f(t + \mid \varepsilon \mid)$

$f(t-0)$ – is the left-hand-side limit of $f(t)$ at a point t,
$f(t+0) = \lim_{\varepsilon \to 0} f(t - \mid \varepsilon \mid)$

$U(\xi)$ – is the Lyapunov matrix; ($U(\xi) = \int_0^\infty K^T(t) W K(t+\xi) dt$)

1 Introduction

This monograph is a summary of the author's research on the determination of the Lyapunov functionals for linear systems with time delay and its applications in the parametric optimization problem.

Lyapunov quadratic functionals are used to test the stability of time delay systems, in computation of critical delay values for time delay systems, in computation of exponential estimates for solutions of time delay systems, in calculation of robustness bounds for uncertain time delay systems, in calculation of a quadratic performance index of quality in the process of parametric optimization for time delay systems.

The stability criteria for time delay systems are formulated in the form of Linear Matrix Inequalities (LMIs). A numerical scheme for construction of Lyapunov functionals was proposed by K. Gu [37]. This method starts with discretization of the Lyapunov functional. The scheme is based on Linear Matrix Inequality (LMI) technique. E. Fridman [30] introduced Lyapunov–Krasovskii functionals for investigation of the stability of linear retarded and neutral type systems with discrete and distributed delays. Method was based on an equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of LMIs. D. Ivanescu et al. [48] proceeded with delay depended stability analysis for linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of LMIs. Q.L. Han [41] obtained a delay-dependent stability criterion for neutral systems with a time-varying discrete delay. This criterion was expressed in the form of LMI and was obtained using the Lyapunov direct method. Q.L. Han [42] investigated robust stability of uncertain neutral systems with discrete and distributed delays, which was based on descriptor model transformation and the decomposition technique, and formulated stability criteria in the form of LMIs. Q.L. Han [43] considered the stability for linear neutral systems with norm-bounded uncertainties in all system matrices and derived a delay-dependent stability criterion. Neither model transformation, nor the bounding technique for cross terms is involved in derivation of the stability criterion. Q.L. Han [44] developed the discretized Lyapunov functional approach to investigation of the stability of linear neutral systems with mixed neutral and discrete delays. Stability criteria, which are applicable to linear neutral systems with both small and no small discrete delays, are formulated in the form of LMIs. Q.L. Han [45] studied the problem of stability of linear time delay systems, both retarded and neutral types, using the discrete delay N-decomposition approach to derive some more general new discrete delay dependent stability criteria. Q.L. Han [46] employed the delay decomposition approach to derive some improved stability criteria for linear neutral systems and to deduce some sufficient conditions for the existence of the Lyapunov functional for a system with k-non-commensurate neutral time delays of a delayed state feedback controller, which ensure asymptotic stability and a prescribed H1 performance level of the corresponding closed-loop system. K. Gu and Y. Liu [38] investigated the stability of coupled differential functional equations using the discretized Lyapunov functional method and set forth the stability condition in the form of LMIs, suitable for numerical computation.

E.F. Infante and W.B. Castelan [47] based the construction of the Lyapunov functional on solution of a matrix differential-difference equation on a finite time interval. V.L. Kharitonov and A.P. Zhabko [66] extended the results of E.F. Infante and W.B. Castelan and proposed a procedure of construction of the quadratic functional for linear retarded type delay systems which could be used for robust stability analysis of time delay systems. This functional was expressed by means of a Lyapunov matrix, which depended on the fundamental matrix of a time delay system. V.L. Kharitonov [50] extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and to neutral type time delay systems with a discrete and distributed delay [52]. V.L. Kharitonov and D. Hinrichsen [62] used the Lyapunov matrix to derive exponential estimates for solutions of exponentially stable time delay systems. V.L. Kharitonov and E. Plischke [65] formulated necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of a retarded system with one delay.

The Lyapunov quadratic functionals are also used to calculation of a value of a quadratic performance index of quality in the process of parametric optimization for time delay systems. The value of that functional at the initial state of the time delay system is equal to the value of a quadratic performance index of quality. To calculate the value of a performance index of quality one needs the formulas for the Lyapunov functional coefficients. For the first time a Lyapunov functional for time delay system was introduced by Yu.M. Repin [79] for the case of a linear system with one retarded-type delay. Yu.M. Repin delivered also the procedure for determination of the functional coefficients. The procedure is as follows. At first it is assumed the form of the functional, afterwords its time derivative on the trajectory of system with a time delay is computed and equated with the negatively definite quadratic form of a system state. In this way we obtain the set of differential and algebraic equations, which enables us to determine the formulas of the functional coefficients. The presented method gives analytical formulas for the coefficients of the Lyapunov functional.

In this monograph the method proposed by Repin [79] is applied to obtain the Lyapunov functionals, with coefficients given by analytical formulas. In Chapter 2. are considered systems with the retarded type time delay. This method is applied to the system with one delay (Chapter 2.2), to the system with two delays (Chapter 2.3), to the retarded type time delay system with both lumped and distributed delay (Chapter 2.4), to the system with a retarded type time-varying delay (Chapter 2.5). In Chapter 3. are considered neutral systems.

Repin's method is applied to the neutral system with lumped delay (Chapter 3.2), to the neutral system with both lumped and distributed delay (Chapter 3.3) and to the neutral system with a time-varying delay (Chapter 3.4). In last years a method of determination of a Lyapunov functional by means of Lyapunov matrices is very popular, see for example [50–66, 72, 73, 76, 81–83]. This method is applied to the parametric optimization problem of retarded type time delay system both with one and two delays (Chapter 4), to the parametric optimization problem of neutral type time delay system for system with one and two delays (Chapter 5). The examples of using of the Lyapunov functionals to calculation of the performance index value in the parametric optimization problem for linear systems with a time delay are given.

2 A linear retarded type time delay system

2.1 Preliminaries

Let us consider a linear system with a retarded type time delay, whose dynamics is described by the equation

$$\begin{cases} \dfrac{dx(t)}{dt} = \mathscr{L}\left(t, x(t), x_t\right) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \varphi \end{cases} \tag{2.1}$$

for $t \geq t_0$, where $x(t) \in \mathbb{R}^n$,$\varphi, x_t \in L^2([-r,0),\mathbb{R}^n)$, $L^2([-r,0),\mathbb{R}^n)$ – is a space of all $\mathbb{R}^n$-valued Lebesgue square integrable functions defined on interval $[-r,0)$ with norm

$$\parallel \varphi \parallel_{L^2} = \sqrt{\int\limits_{-r}^{0} \left(\parallel \varphi(t) \parallel_{\mathbb{R}^n}^2\right) dt}$$

.

The function $\mathscr{L}$ is a linear and continuous and defined on the space

$$[0,\infty) \times \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n)$$

$$\mathscr{L} : [0,\infty) \times \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n) \to \mathbb{R}^n \tag{2.2}$$

The space of initial values of system (2.1) is given by the Cartesian product

$$\mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n)$$

The norm of an initial value (x_0, φ) is given by

$$\parallel (x_0, \varphi) \parallel_{\mathbb{R}^n \times L^2} = \sqrt{\parallel x_0 \parallel_{\mathbb{R}^n}^2 + \parallel \varphi \parallel_{L^2}^2} \tag{2.3}$$

The solution of the functional-differential equation (2.1) with initial value (x_0, φ) or simply a solution through (x_0, φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$.

$$x(\cdot, t_0, (x_0, \varphi)) \in W^{1,2}([t_0, \infty), \mathbb{R}^n) \tag{2.4}$$

Definition 2.1. *The function $x_t(t_0, (x_0, \varphi)) : [-r, 0) \to \mathbb{R}^n$ is called a **shifted restriction** of $x(\cdot, t_0, (x_0, \varphi))$ to an interval $[t-r, t)$ and is defined by a formula*

$$x_t(t_0, (x_0, \varphi))(\theta) := x(t+\theta, t_0, (x_0, \varphi)) \tag{2.5}$$

for $t \geq t_0$ and $\theta \in [-r, 0)$.

When $t_0 = 0$, the shifted restriction is denoted as $x_t(x_0, \varphi)$. When initial condition is established, the shifted restriction is denoted by x_t.
The state of system (2.1) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \tag{2.6}$$

for $t \geq t_0$.
The state space is defined by the formula

$$X = \mathbb{R}^n \times L^2([-r, 0), \mathbb{R}^n) \tag{2.7}$$

with norm given by the term

$$\| (x_0, x_t) \|_{\mathbb{R}^n \times L^2} = \sqrt{\| x_0 \|_{\mathbb{R}^n}^2 + \| x_t \|_{L^2}^2} \tag{2.8}$$

We assume that system (2.1) admits the trivial solution, i.e., the following identity holds:

$$\mathscr{L}(t, 0_{\mathbb{R}^n}, 0_{L^2}) \equiv 0$$

for $t \geq 0$.
Let $x(t, t_0, (x_0, \varphi))$ be the solution of system (2.1) with initial condition (x_0, φ) for $t \geq t_0$.

Definition 2.2. [56] *The trivial solution of system (2.1) is said to be **stable** if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that for every $(x_0, \varphi) \in \mathbb{R}^n \times L^2([-r, 0), \mathbb{R}^n)$*

$$\| (x_0, \varphi) \|_{\mathbb{R}^n \times L^2} \leq \delta(\varepsilon, t_0) \Rightarrow \| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \leq \varepsilon$$

for every $t \geq t_0$.

*If $\delta(\varepsilon, t_0)$ can be chosen independently of t_0, then the trivial solution is said to be **uniformly stable.***

Definition 2.3. [56] *The trivial solution of system (2.1) is said to be **asymptotically stable** if it is stable and $\| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \to 0$ as $t - t_0 \to \infty$.*

Definition 2.4. [56] *The trivial solution of system (2.1) is said to be* ***exponentially stable*** *if there exist* $\delta > 0$*,* $M \geq 1$ *and* $\sigma > 0$ *such that for every* $t_0 \geq 0$ *and initial condition* $(x_0, \varphi) \in \mathbb{R}^n \times L^2([-r, 0), \mathbb{R}^n)$*, with* $\| (x_0, \varphi) \|_{\mathbb{R}^n \times L^2} \leq \delta$ *the following inequality holds*

$$\| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \leq M e^{-\sigma(t-t_0)} \| (x_0, \varphi) \|_{\mathbb{R}^n \times L^2}$$

for every $t \geq t_0$*.*

In a parametric optimization problem will be used an integral quadratic performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t) W x(t) dt \tag{2.9}$$

where $W \in \mathbb{R}^{n \times n}$ is a positive definite matrix.

Definition 2.5. [16,18] *The functional* $V : X \times [t_0, \infty) \to \mathbb{R}$ *is* ***positive definite*** *if it is continuous and there exists a positive definite functional* $H : X \to \mathbb{R}$ *such that* $V(x,t) \geq H(x)$ *and* $V(0,t) = H(0) = 0$ *for* $x \in X$ *and* $t \geq t_0$*.*

Definition 2.6. [16, 18] *A positive definite functional* $V : X \times [t_0, \infty) \to \mathbb{R}$ *is* ***upper bounded*** *if there exists a positive definite functional* $H : X \to \mathbb{R}$ *such that* $V(x,t) \leq H(x)$ *for* $x \in X$ *and* $t \geq t_0$*.*

Definition 2.7. [16, 18] *A time derivative of the functional* $V(x(t), x_t, t)$ *at* $(x(t_0), \varphi, t_0)$ *on the trajectory of system (2.1) is given by the formula*

$$\frac{dV(x(t_0), \varphi, t_0)}{dt} = \limsup_{h \to 0} \frac{1}{h} \left[V\Big(x(t_0 + h), x_{t_0+h}, t_0 + h\Big) - V\Big(x(t_0), \varphi, t_0\Big) \right] \tag{2.10}$$

Definition 2.8. [16, 18] *The functional* $V : X \times [t_0, \infty) \to \mathbb{R}$ *is called a* ***Lyapunov functional*** *if*

1. V is a positive definite upper bounded functional

2. V is differentiable

3. A time derivative of V computed according to the formula (2.10) on the trajectory of system (2.1) is negative definite

From the assumption that the Lyapunov functional is upper bounded results that there exists a functional H such that

$$0 \leq V(x(t), x_t, t) \leq H(x(t), x_t) \tag{2.11}$$

for $t \geq t_0$.

We had assumed that system (2.1) admits the trivial solution, i.e., the following identity holds:

$$\mathscr{L}(t, 0_{\mathbb{R}^n}, 0_{L^2}) \equiv 0$$

for $t \geq 0$.
When the system (2.1) is asymptotically stable $\lim_{t\to\infty} H(x(t), x_t) = 0$ implies $\lim_{t\to\infty} V(x(t), x_t, t) = 0$.
Hence

$$\int_{t_0}^{\infty} \frac{dV(x(t), x_t, t)}{dt} dt = \lim_{t\to\infty} V(x(t), x_t, t) - \lim_{t\to t_0} V(x(t), x_t, t) =$$

$$= -V(\lim_{t\to t_0}(x(t), x_t, t)) = -V(x(t_0), \varphi, t_0) \tag{2.12}$$

Assume that the time derivative of the Lyapunov functional V is given as a quadratic form

$$\frac{dV(x(t), x_t, t)}{dt} \equiv -x^T(t) W x(t) \tag{2.13}$$

for $t \geq t_0$, where $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix.
It follows from (2.9) and (2.13) that

$$J = \int_{t_0}^{\infty} x^T(t) W x(t) dt = V(x(t_0), \varphi, t_0) \tag{2.14}$$

Corollary 2.1. *If one constructs a positive definite functional such that its time derivative computed on the trajectory of system (2.1) is given as a negative definite quadratic form (2.13) one can not only investigate the system (2.1) stability but also calculate a value of a square indicator of quality (2.9) of the parametric optimization problem.*

In derivation of the formula (2.14) we had assumed that system (2.1) was asymptotically stable to achieve convergence of the integral (2.12). Existence of the Lyapunov functional is the sufficient condition for asymptotically stability. When we construct a functional which is positive definite, its time derivative on the trajectory of dynamical system is negative definite and value of this functional depends on the value of the controller parameter , we can determine the region of stability. The system is asymptotically stable for this controller parameters for which the value of the functional is positive, when the value is negative system becomes unstable. The value of the controller parameter for which the functional changes the sign of value from positive to negative, is the critical value of the controller parameter. In the stability region we can search for optimal value of controller parameter which minimizes the index of quality. The optimization procedure will be made by means of Matlab *fminsearch* function.

2.2 The Lyapunov functional for a linear system with one delay

2.2.1 Mathematical model of a linear time delay system with one delay

Let us consider a linear system with a retarded type time delay whose dynamics is described by a functional-differential equation (FDE)

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx(t-r) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{2.15}$$

$t \geq t_0$, $\theta \in [-r,0)$, $r \geq 0$, $A,B \in \mathbb{R}^{n\times n}$, $x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $\varphi \in L^2([-r,0),\mathbb{R}^n)$. The space of initial data is given by the Cartesian product $\mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n)$.
The norm of an initial value (x_0,φ) is given by (2.3).
The solution of the functional-differential equation (2.15) with initial value (x_0,φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$.

$$x(\cdot,t_0,(x_0,\varphi)) \in W^{1,2}([t_0,\infty),\mathbb{R}^n) \tag{2.16}$$

One can obtain a solution of FDE (2.15) using a step method. The step method is a basic method for solving FDE with a lumped delay. A solution is found on successive intervals, one after another, by solving an ordinary equation without delay in each interval.
For $t \in [t_0,\, t_0+r]$ the equation (2.15) takes a form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + B\varphi(t-r) \\ x(t_0) = x_0 \end{cases} \tag{2.17}$$

The solution of equation (2.17) is given by a term

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\varphi(\tau-r)d\tau \tag{2.18}$$

$$\Psi(t) = x(t) \tag{2.19}$$

$$x(t_0+r) = x_1 \tag{2.20}$$

For $t \in [t_0+r,\, t_0+2r]$ the equation (2.15) takes a form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + B\Psi(t-r) \\ x(t_0+r) = x_1 \end{cases} \tag{2.21}$$

and so on. By means of this procedure one can construct the solution in any finite interval.

One can write the equation (2.15) in the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx_t(-r) \\ x(t_0) = x_0 \\ x_{t_0} = \varphi \in L^2([-r,0),\mathbb{R}^n) \end{cases} \tag{2.22}$$

for $t \geq t_0$. Where $x_t \in L^2([-r,0),\mathbb{R}^n)$ is a shifted restriction of $x(\cdot,t_0,(x_0,\varphi))$ to the segment $[t-r,t)$.
There holds a relationship

$$x_{t_0}(\cdot,t_0,(x_0,\varphi)) = \varphi \tag{2.23}$$

where $x_{t_0}(\cdot,t_0,(x_0,\varphi))$ is a shifted restriction of $x(\cdot,t_0,(x_0,\varphi))$ to an interval $[t_0-r,t_0)$.
The theorems of existence, continuous dependence and uniqueness of solutions of equation (2.22) are given in [32]. The controllability of the systems with time delay is presented in [67].
The state of system (2.22) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \tag{2.24}$$

for $t \geq t_0$ where $x(t) \in \mathbb{R}^n$, $x_t \in L^2([-r,0),\mathbb{R}^n)$.
The state space is defined by a formula

$$X = \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n) \tag{2.25}$$

$S = 0$ is an equilibrium point of the system (2.22).
In a parametric optimization problem will be used an integral quadratic performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t)x(t)dt \tag{2.26}$$

The value of the performance index of quality (2.26) is given by the term (2.14), which for system (2.22) takes a form

$$J = \int_{t_0}^{\infty} x^T(t)x(t)dt = V(x_0,\varphi) \tag{2.27}$$

To calculate the value of the performance index (2.27), which is equal to the value of the Lyapunov functional at the initial state of system (2.22), one needs a mathematical formula of that functional.

2.2.2 Determination of the Lyapunov functional

On the state space X we define a quadratic functional V positive definite, differentiable, given by the formula [79]

$$V(x(t),x_t) = x^T(t)\alpha x(t) + \int_{-r}^{0} x^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \quad (2.28)$$

for $t \geq t_0$, where $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in C^1([-r,0],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$
$\Omega = \{(\theta,\sigma): \theta \in [-r,0], \sigma \in [\theta,0]\}$, C^1 is a space of all continuous functions with continuous derivative.
In this paragraph will be given a procedure of determination of the functional (2.28) coefficients to obtain the Lyapunov functional.
In calculation of the time derivative of the functional (2.28) will be used the following Lemma.

Lemma 2.1. *There holds the relationship*

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \quad (2.29)$$

Proof.

$$x_t(\theta) = x(t+\theta) \ \text{ for } \ t \geq t_0,\ \theta \in [-r,0)$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x(t+\theta)}{\partial t} = \frac{\partial x(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial x(\xi)}{\partial \xi} \ \text{ for } \ \xi = t+\theta$$

$$\frac{\partial x_t(\theta)}{\partial \theta} = \frac{\partial x(t+\theta)}{\partial \theta} = \frac{\partial x(\xi)}{\partial \xi}\frac{\partial \xi}{\partial \theta} = \frac{\partial x(\xi)}{\partial \xi} \ \text{ for } \ \xi = t+\theta$$

hence

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta}$$

□

The time derivative of the functional (2.28) on the trajectory of system (2.22) is computed. This time derivative is defined by the formula (2.10) which for system (2.22) takes a form

$$\frac{dV(x(t_0),\varphi)}{dt} = \limsup_{h\to 0}\frac{1}{h}\left[V\left(x(t_0+h),x_{t_0+h}\right) - V(x(t_0),\varphi)\right] \quad (2.30)$$

It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (2.28) and one substitutes in place of $dx(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dx(t)}{dt} = Ax(t) + Bx_t(-r) \quad (2.31)$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \quad (2.32)$$

In such a manner one attains [79]

$$\frac{dV(x(t),x_t)}{dt} = x^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2}\right]x(t) +$$
$$+ x^T(t)\left[2\alpha B - \beta(-r)\right]x_t(-r) +$$
$$+ \int_{-r}^{0} x^T(t)\left[A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0)\right]x_t(\theta)d\theta +$$
$$+ \int_{-r}^{0} x_t^T(-r)\left[B^T\beta(\theta) - \delta(-r,\theta)\right]x(t+\theta)d\theta +$$
$$- \int_{-r}^{0}\int_{\theta}^{0} x_t^T(\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta \tag{2.33}$$

To achieve negative definiteness of that derivative we assume that

$$\frac{dV(x(t),x_t)}{dt} \equiv -x^T(t)x(t) \tag{2.34}$$

From equations (2.2.2) and (2.34) we obtain the set of equations

$$A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} = -I \tag{2.35}$$

$$2\alpha B - \beta(-r) = 0 \tag{2.36}$$

$$A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0) = 0 \tag{2.37}$$

$$B^T\beta(\theta) - \delta(-r,\theta) = 0 \tag{2.38}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} = 0 \tag{2.39}$$

for $\theta \in [-r,0]$, $\sigma \in [-r,0]$.

The solution of a differential equation (2.39) is given in the form

$$\delta(\theta,\sigma) = f(\theta-\sigma) \tag{2.40}$$

where $f \in C^1([-r,r],\mathbb{R}^{n\times n})$.

From equations (2.40) and (2.38) one obtains

$$\delta(-r,\theta) = f(-r-\theta) = B^T\beta(\theta) \tag{2.41}$$

$$f(\theta) = B^T\beta(-r-\theta) \tag{2.42}$$

$$\delta^T(\theta,0) = f^T(\theta) = \beta^T(-r-\theta)B \tag{2.43}$$

After putting (2.43) into (2.37) one attains a formula

$$\frac{d\beta(\theta)}{d\theta} = A^T\beta(\theta) + \beta^T(-r-\theta)B$$

for $\theta \in [-r,0]$.
The derivative of the function $\beta(-\theta - r)$ with respect to θ is calculated

$$\frac{d\beta(-r-\theta)}{d\theta} = \frac{d\beta(\xi)}{d\xi}\frac{d\xi}{d\theta} = -\frac{d\beta(\xi)}{d\xi} =$$
$$= -A^T\beta(\xi) - \beta^T(-r-\xi)B = -A^T\beta(-r-\theta) - \beta^T(\theta)B \tag{2.44}$$

where

$$\xi = -r - \theta \tag{2.45}$$

The set of the differential equations is obtained

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} = A^T\beta(\theta) + \beta^T(-r-\theta)B \\ \dfrac{d\beta(-r-\theta)}{d\theta} = -A^T\beta(-r-\theta) - \beta^T(\theta)B \end{cases} \tag{2.46}$$

for $\theta \in [-r,0]$.
A new function is introduced

$$\kappa(\theta) = \beta^T(-\theta - r) \tag{2.47}$$

for $\theta \in [-r,0]$.
The set of the differential equations (2.46) takes a form

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} = A^T\beta(\theta) + \kappa(\theta)B \\ \dfrac{d\kappa(\theta)}{d\theta} = -\kappa(\theta)A - B^T\beta(\theta) \end{cases} \tag{2.48}$$

for $\theta \in [-r,0]$ with initial conditions $\beta(-r)$ and $\kappa(-r)$.
Using the Kronecker product the set of differential equations (2.48) can be reshape to the form

$$\begin{bmatrix} \dfrac{d}{d\theta} col\beta(\theta) \\ \dfrac{d}{d\theta} col\kappa(\theta) \end{bmatrix} = \begin{bmatrix} I \otimes A^T & B^T \otimes I \\ -I \otimes B^T & -A^T \otimes I \end{bmatrix} \begin{bmatrix} col\beta(\theta) \\ col\kappa(\theta) \end{bmatrix} \tag{2.49}$$

for $\theta \in [-r,0]$ with initial conditions $col\beta(-r)$ and $col\kappa(-r)$.
The solution of initial value problem (2.49) has a form

$$\begin{bmatrix} col\beta(\theta) \\ col\kappa(\theta) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\theta+r) & \Phi_{12}(\theta+r) \\ \Phi_{21}(\theta+r) & \Phi_{22}(\theta+r) \end{bmatrix} \begin{bmatrix} col\beta(-r) \\ col\kappa(-r) \end{bmatrix} \tag{2.50}$$

where a matrix $\Phi(\theta) = \begin{bmatrix} \Phi_{11}(\theta) & \Phi_{12}(\theta) \\ \Phi_{21}(\theta) & \Phi_{22}(\theta) \end{bmatrix}$ is a fundamental matrix of system (2.49).
Equation (2.50) implies

$$col\beta(\theta)\,|_{\theta=-\frac{r}{2}} = \Phi_{11}\left(\frac{r}{2}\right) col\beta(-r) + \Phi_{12}\left(\frac{r}{2}\right) col\kappa(-r) \tag{2.51}$$

$$col\kappa(\theta)\,|_{\theta=-\frac{r}{2}} = \Phi_{21}\left(\frac{r}{2}\right) col\beta(-r) + \Phi_{22}\left(\frac{r}{2}\right) col\kappa(-r) \tag{2.52}$$

Equation (2.47) implies

$$\beta^T(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \tag{2.53}$$

Formula (2.53) presents the algebraic linear relationship between initial conditions $col\beta(-r)$ and $col\kappa(-r)$.
Equation (2.47) implies

$$\kappa(-r) = \beta^T(0) \tag{2.54}$$

Formula (2.35) takes a form

$$A^T\alpha + \alpha A + \frac{\kappa(-r) + \kappa^T(-r)}{2} = -I \tag{2.55}$$

Formulas (2.55), (2.36) and (2.53) create the set of algebraic equations

$$\begin{cases} A^T\alpha + \alpha A + \dfrac{\kappa(-r) + \kappa^T(-r)}{2} = -I \\ 2\alpha B - \beta(-r) = 0 \\ \beta^T(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \end{cases} \tag{2.56}$$

The set of algebraic equations (2.56) allows for determination of the matrix α and the initial conditions of the set of differential equations (2.49).
From equations (2.42) and (2.47) one attains

$$f(\theta) = B^T\beta(-r-\theta) = B^T\kappa^T(\theta) \tag{2.57}$$

for $\theta \in [-r, 0]$.
Taking into account (2.40) and (2.57) one obtains

$$\delta(\theta,\sigma) = B^T\kappa^T(\theta-\sigma) \tag{2.58}$$

In this way one obtained all coefficients of the functional (2.28). This coefficients depend on the matrices A and B of system (2.15). The time derivative of the functional (2.28) is negative definite.

2.2.3 The examples

2.2.3.1 Inertial system with delay and a P controller

Let us consider a first order inertial system with delay described by equation [8]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{q}{T}x(t) + \dfrac{k_0}{T}u(t-r) \\ x(0) = x_o \\ x(\theta) = 0 \\ u(t) = -px(t) \end{cases} \tag{2.59}$$

$t \geq 0$, $x(t) \in \mathbb{R}$, $\theta \in [-r, 0)$, p, k_0, T, q, $x_0 \in \mathbb{R}$, $r \geq 0$. The parameter k_0 is a gain of a plant, p is a gain of a P controller, T is a system time constant, x_0 is an initial state of a system. In the case $q = 1$ an equation (2.59) describes a static object and in the case $q = 0$ an equation (2.59) describes an astatic object.
One can reshape an equation (2.59) to the form

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{q}{T}x(t) - \dfrac{k_0 p}{T}x(t-r) \\ x(0) = x_o \\ x(\theta) = 0 \end{cases} \tag{2.60}$$

for $t \geq 0$.
One searches for a parameter k whose minimize an integral quadratic performance index

$$J = \int_0^\infty x^2(t)dt \tag{2.61}$$

The Lyapunov functional V is defined by the formula

$$V(x(t), x(t+\cdot)) = \alpha x^2(t) + \int_{-r}^{0} x(t)\beta(\theta)x(t+\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} x(t+\theta)\delta(\theta,\sigma)x(t+\sigma)d\sigma d\theta$$

According to a term (2.27) a performance index value is given by the formula

$$J = \int_0^\infty x^2(t)dt = V(x(t), x(t+\cdot)) \mid_{t=0} \tag{2.62}$$

The set of a differential equation (2.48) takes a form

$$\begin{bmatrix} \dfrac{d\beta(\theta)}{d\theta} \\ \dfrac{d\kappa(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} -\dfrac{q}{T} & -\dfrac{k_0 p}{T} \\ \dfrac{k_0 p}{T} & \dfrac{q}{T} \end{bmatrix} \begin{bmatrix} \beta(\theta \\ \kappa(\theta) \end{bmatrix} \tag{2.63}$$

The fundamental matrix of system (2.63) takes a form

$$R(\theta)=\begin{bmatrix}\cosh\lambda\theta-\dfrac{q}{T\lambda}\sinh\lambda\theta & -\dfrac{k_0p}{T\lambda}\sinh\lambda\theta\\ \dfrac{k_0p}{T\lambda}\sinh\lambda\theta & \cosh\lambda\theta+\dfrac{q}{T\lambda}\sinh\lambda\theta\end{bmatrix}\tag{2.64}$$

where

$$\lambda=\frac{\sqrt{q^2-k_0^2p^2}}{T}\tag{2.65}$$

The set of algebraic equations (2.56) takes a form

$$\begin{cases}-2\dfrac{q}{T}\alpha+\kappa(-r)=-1\\ 2\alpha\dfrac{k_0p}{T}+\beta(-r)=0\\ \left[\cosh\dfrac{\lambda r}{2}-\dfrac{q+k_0p}{T\lambda}\sinh\dfrac{\lambda r}{2}\right]\beta(-r)+\left[-\cosh\dfrac{\lambda r}{2}-\dfrac{q+k_0p}{T\lambda}\sinh\dfrac{\lambda r}{2}\right]\kappa(-r)=0\end{cases}\tag{2.66}$$

From an equation (2.66) one obtains a parameter α and the initial conditions of the differential equation (2.63).

$$\alpha=\frac{\cosh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T\lambda}\sinh\dfrac{\lambda r}{2}}{2\left(\lambda\sinh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T}\cosh\dfrac{\lambda r}{2}\right)}\tag{2.67}$$

$$\beta(-r)=\frac{\dfrac{k_0p}{T}\left(\cosh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T\lambda}\sinh\dfrac{\lambda r}{2}\right)}{\lambda\sinh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T}\cosh\dfrac{\lambda r}{2}}\tag{2.68}$$

$$\kappa(-r)=\frac{-\dfrac{k_0p}{T}\left(\cosh\dfrac{\lambda r}{2}-\dfrac{q+k_0p}{T\lambda}\sinh\dfrac{\lambda r}{2}\right)}{\lambda\sinh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T}\cosh\dfrac{\lambda r}{2}}\tag{2.69}$$

Having a fundamental matrix (2.64) and the initial conditions of the differential equation (2.63) one obtains

$$\beta(\theta)=\frac{k_0p}{T\left(\lambda\sinh\dfrac{\lambda r}{2}+\dfrac{q+k_0p}{T}\cosh\dfrac{\lambda r}{2}\right)}\left[\left(\frac{q+k_0p}{T\lambda}\cosh\frac{\lambda r}{2}-\sinh\frac{\lambda r}{2}\right)\sinh\lambda\theta+\right.$$
$$\left.+\left(\frac{q+k_0p}{T\lambda}\sinh\frac{\lambda r}{2}-\cosh\frac{\lambda r}{2}\right)\cosh\lambda\theta\right]\tag{2.70}$$

$$\kappa(\theta) = -\frac{k_0 p}{T\lambda}\sinh\lambda\theta +$$

$$-\frac{k_0 p}{T\left(\lambda\sinh\frac{\lambda r}{2} + \frac{q+k_0 p}{T}\cosh\frac{\lambda r}{2}\right)}\left(\cosh\frac{\lambda r}{2} + \frac{q+k_0 p}{T\lambda}\sinh\frac{\lambda r}{2}\right)\cosh\lambda\theta \tag{2.71}$$

$$\delta(\theta,\sigma) = \frac{k_0^2 p^2}{T^2\lambda}\sinh\lambda(\theta-\sigma) +$$

$$+\frac{k_0^2 p^2}{T^2\left(\lambda\sinh\frac{\lambda r}{2} + \frac{q+k_0 p}{T}\cosh\frac{\lambda r}{2}\right)}\left(\cosh\frac{\lambda r}{2} + \frac{q+k_0 p}{T\lambda}\sinh\frac{\lambda r}{2}\right)\cosh\lambda(\theta-\sigma) \tag{2.72}$$

Now a performance index value is calculated

$$J = \frac{x_0^2}{2\left(\lambda\sinh\frac{\lambda r}{2} + \frac{q+k_0 p}{T}\cosh\frac{\lambda r}{2}\right)}\left[\cosh\frac{\lambda r}{2} + \frac{q+k_0 p}{T\lambda}\sinh\frac{\lambda r}{2}\right] \tag{2.73}$$

Figure 2.1 shows the value of the index $J(p)$ for $x_0 = 1$, $k_0 = 1$, $q = 1$, and $T = 5$ and $r = 2$. You can see that there exists a critical value of the gain p_{crit}. The system (2.60) is stable for gains less then critical one and unstable for gains grater then critical.

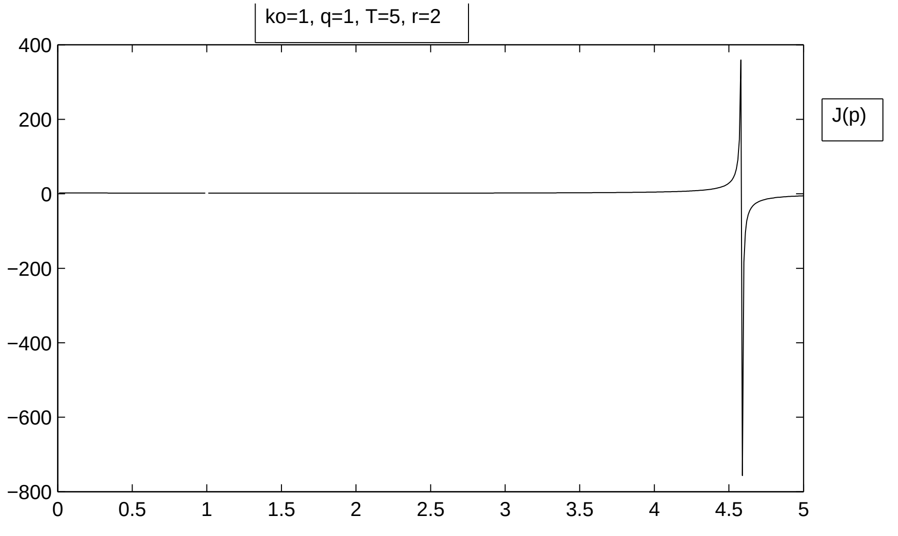

Fig. 2.1. Value of the index $J(p)$ for p greater then p_{crit}

Figure 2.2 shows the value of the index $J(p)$ for p less then critical gain. You can see that the function $J(p)$ is convex and has a minimum.

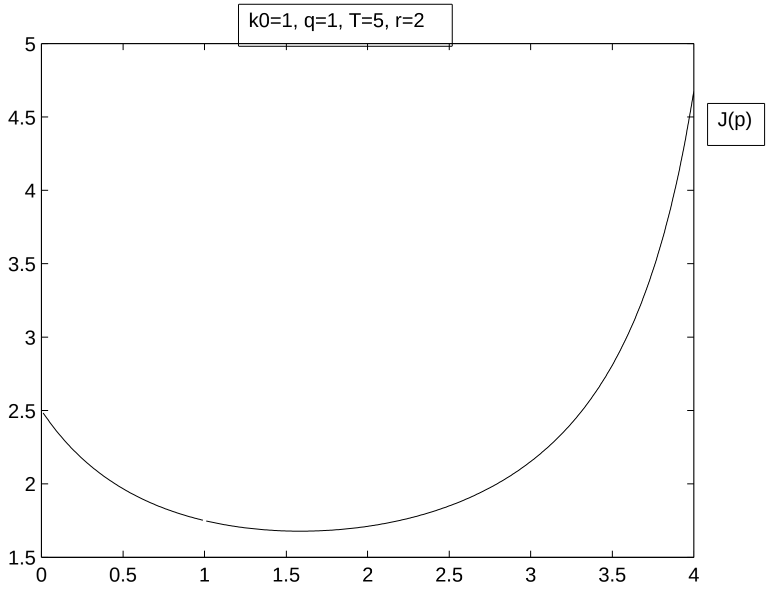

Fig. 2.2. Value of the index $J(p)$ for p less then p_{crit}

We search for an optimal gain which minimizes the index (2.73). Optimization results, obtained by means of the Matlab function *fminsearch*, are given in Table 2.1. These results are obtained for $x_0 = 1$, $k_0 = 1$, $q = 1$, and $T = 5$.

Table 2.1
Optimization results

Delay r	Optimal gain	Index value	Critical gain
1.0	3.4329	1.1014	8.50
1.5	2.2020	1.4331	5.89
2.0	1.5871	1.6778	4.58
2.5	1.2188	1.8610	3.80
3.0	0.9738	1.9997	3.28
3.5	0.7993	2.1060	2.91
4.0	0.6689	2.1880	2.64
4.5	0.5681	2.2518	2.43
5.0	0.4878	2.3019	2.26

2.2.3.2 Inertial system with delay and an I controller

Let us consider a first order inertial system with delay described by the equation [9]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{1}{T}x(t) + \dfrac{k_0}{T}u(t-r) \\ x(0) = x_o \\ x(\theta) = 0 \\ u(t) = -\dfrac{1}{T_i}\displaystyle\int_0^t x(\xi)d\xi + z_0 \end{cases} \tag{2.74}$$

$t \geq 0$, $x(t) \in \mathbb{R}$, $\theta \in [-r, 0)$, T_i, k_0, T, x_0, $z_0 \in \mathbb{R}$, $r \geq 0$. The parameter k_0 is a gain of a plant, T_i is a time of isodrome of an I controller, T is a system time constant, x_0 is an initial state of a system, z_0 is an amplitude of a disturbance.

One introduces the state variables $x_1(t)$ and $x_2(t)$ as follows

$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = \dfrac{1}{T_i}\displaystyle\int_0^t x(\xi)d\xi \end{cases} \tag{2.75}$$

The set of equations (2.74) takes a form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -\dfrac{1}{T}x_1(t) + \dfrac{k_0}{T}u(t-r) \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{T_i}x_1(t) \\ x_1(0) = x_o \\ x_2(0) = 0 \\ x_1(\theta) = 0 \\ x_2(\theta) = 0 \\ u(t) = -x_2(t) + z_0 \end{cases} \tag{2.76}$$

for $t \geq 0$, $\theta \in [-r, 0)$.

One can reshape equation (2.76) to the form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -\dfrac{1}{T}x_1(t) - \dfrac{k_0}{T}x_2(t-r) + \dfrac{k_0 z_0}{T} \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{T_i}x_1(t) \\ x_1(0) = x_o \\ x_2(0) = 0 \\ x_1(\theta) = 0 \\ x_2(\theta) = 0 \end{cases} \tag{2.77}$$

for $t \geq 0$, $\theta \in [-r, 0)$.

The equilibrium point of system (2.77) is given by a term

$$\begin{cases} x_1^* = 0 \\ x_2^* = z_0 \end{cases} \tag{2.78}$$

One introduces a new variable

$$\begin{cases} y_1(t) = x_1(t) \\ y_2(t) = x_2(t) - z_0 \end{cases} \tag{2.79}$$

Taking a term (2.79) into account a set of equations (2.77) takes a form

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -\dfrac{1}{T}y_1(t) - \dfrac{k_0}{T}y_2(t-r) \\ \dfrac{dy_2(t)}{dt} = \dfrac{1}{T_i}y_1(t) \\ y_1(0) = x_o \\ y_2(0) = -z_0 \\ y_1(\theta) = 0 \\ y_2(\theta) = -z_0 \end{cases} \tag{2.80}$$

Equations (2.80) in a matrix form are as below

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + By(t-r) \\ y(0) = \begin{bmatrix} x_0 \\ -z_0 \end{bmatrix} \\ y(\theta) = \begin{bmatrix} 0 \\ -z_0 \end{bmatrix} \end{cases} \tag{2.81}$$

where

$$A = \begin{bmatrix} -\dfrac{1}{T} & 0 \\ \dfrac{1}{T_i} & 0 \end{bmatrix} \tag{2.82}$$

$$B = \begin{bmatrix} 0 & -\dfrac{k_0}{T} \\ 0 & 0 \end{bmatrix} \tag{2.83}$$

One searches for a parameter T_i whose minimize an integral quadratic performance index

$$J = \int_0^\infty y^T(t)y(t)dt \tag{2.84}$$

The Lyapunov functional is given

$$V(y(t),y(t+\cdot)) = y^T(t)\alpha y(t) + \int_{-r}^{0} y^T(t)\beta(\theta)y(t+\theta)d\theta +$$

$$+\int_{-r}^{0}\int_{\theta}^{0} y^T(t+\theta)\delta(\theta,\sigma)y(t+\sigma)d\sigma d\theta \tag{2.85}$$

where

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \tag{2.86}$$

$$\beta(\theta) = \begin{bmatrix} \beta_{11}(\theta) & \beta_{12}(\theta) \\ \beta_{12}(\theta) & \beta_{22}(\theta) \end{bmatrix} \tag{2.87}$$

$$\delta(\theta,\sigma) = \begin{bmatrix} \delta_{11}(\theta,\sigma) & \delta_{12}(\theta,\sigma) \\ \delta_{21}(\theta,\sigma) & \delta_{22}(\theta,\sigma) \end{bmatrix} \tag{2.88}$$

$$J = \int_{0}^{\infty} y^T(t)y(t)dt = V(y(0),y(\theta)) \tag{2.89}$$

The set of differential equations (2.48) takes a form

$$\begin{cases} \dfrac{d\beta_{11}(\theta)}{d\theta} = -\dfrac{1}{T}\beta_{11}(\theta) + \dfrac{1}{T_i}\beta_{21}(\theta) \\ \dfrac{d\beta_{21}(\theta)}{d\theta} = 0 \\ \dfrac{d\beta_{12}(\theta)}{d\theta} = -\dfrac{1}{T}\beta_{12}(\theta) + \dfrac{1}{T_i}\beta_{22}(\theta) - \dfrac{k_0}{T}\kappa_{11}(\theta) \\ \dfrac{d\beta_{22}(\theta)}{d\theta} = -\dfrac{k_0}{T}\kappa_{12}(\theta) \\ \dfrac{d\kappa_{11}(\theta)}{d\theta} = \dfrac{1}{T}\kappa_{11}(\theta) - \dfrac{1}{T_i}\kappa_{21}(\theta) \\ \dfrac{d\kappa_{21}(\theta)}{d\theta} = 0 \\ \dfrac{d\kappa_{12}(\theta)}{d\theta} = \dfrac{1}{T}\kappa_{12}(\theta) - \dfrac{1}{T_i}\kappa_{22}(\theta) + \dfrac{k_0}{T}\beta_{11}(\theta) \\ \dfrac{d\kappa_{22}(\theta)}{d\theta} = \dfrac{k_0}{T}\beta_{12}(\theta) \end{cases} \tag{2.90}$$

for $\theta \in [-r,0]$, where

$$\kappa(\theta) = \beta(-\theta - r) \tag{2.91}$$

for $\theta \in [-r,0]$.

The two first equations of (2.56) takes a form

$$\begin{cases} -\dfrac{2}{T}\alpha_{11}+\dfrac{2}{T_i}\alpha_{12}+\kappa_{11}(-r)=-1 \\ -\dfrac{2}{T}\alpha_{12}+\dfrac{2}{T_i}\alpha_{22}+\kappa_{12}(-r)+\kappa_{21}(-r)=0 \\ \kappa_{22}(-r)=-1 \\ \beta_{11}(-r)=0 \\ \beta_{21}(-r)=0 \\ -\dfrac{2k_0}{T}\alpha_{11}-\beta_{12}(-r)=0 \\ -\dfrac{2k_0}{T}\alpha_{12}-\beta_{22}(-r)=0 \end{cases} \tag{2.92}$$

Equations (2.90), (2.91) and (2.92) implies $\beta_{21}(\theta)=0$, $\kappa_{21}(\theta)=0$, $\beta_{11}(\theta)=0$, $\kappa_{11}(\theta)=0$ for $\theta\in[-r,0]$.
Formula (2.87) takes a form

$$\beta(\theta)=\begin{bmatrix} 0 & \beta_{12}(\theta) \\ 0 & \beta_{22}(\theta) \end{bmatrix} \tag{2.93}$$

and

$$\kappa(\theta)=\begin{bmatrix} 0 & \kappa_{12}(\theta) \\ 0 & \kappa_{22}(\theta) \end{bmatrix} \tag{2.94}$$

The set of equations (2.90) takes a form

$$\begin{cases} \dfrac{d\beta_{12}(\theta)}{d\theta}=-\dfrac{1}{T}\beta_{12}(\theta)+\dfrac{1}{T_i}\beta_{22}(\theta) \\ \dfrac{d\beta_{22}(\theta)}{d\theta}=-\dfrac{k_0}{T}\kappa_{12}(\theta) \\ \dfrac{d\kappa_{12}(\theta)}{d\theta}=\dfrac{1}{T}\kappa_{12}(\theta)-\dfrac{1}{T_i}\kappa_{22}(\theta) \\ \dfrac{d\kappa_{22}(\theta)}{d\theta}=\dfrac{k_0}{T}\beta_{12}(\theta) \end{cases} \tag{2.95}$$

The fundamental matrix of solutions of equation (2.95) is given by

$$R(\theta)=\frac{1}{s_1^2+s_2^2}\begin{bmatrix} r_{11}(\theta) & r_{12}(\theta) & r_{13}(\theta) & r_{14}(\theta) \\ r_{21}(\theta) & r_{22}(\theta) & r_{23}(\theta) & r_{24}(\theta) \\ r_{31}(\theta) & r_{32}(\theta) & r_{33}(\theta) & r_{34}(\theta) \\ r_{41}(\theta) & r_{42}(\theta) & r_{43}(\theta) & r_{44}(\theta) \end{bmatrix} \tag{2.96}$$

where

$$s_i=\frac{1}{T}\sqrt{\frac{\sqrt{1+\dfrac{4k_0^2T^2}{T_i^2}}+(-1)^i}{2}} \quad \text{for} \quad i=1,2 \tag{2.97}$$

$$r_{11}(\theta) = s_1^2 \cos s_1\theta - \frac{s_1}{T} \sin s_1\theta + s_2^2 \cosh s_2\theta - \frac{s_2}{T} \sinh s_2\theta \tag{2.98}$$

$$r_{21}(\theta) = \frac{k_0^2}{T^2 T_i} \left(-\frac{1}{s_1} \sin s_1\theta + \frac{1}{s_2} \sinh s_2\theta \right) \tag{2.99}$$

$$r_{31}(\theta) = \frac{k_0}{T T_i} \Big(\cos s_1\theta - \cosh s_2\theta \Big) \tag{2.100}$$

$$r_{41}(\theta) = \frac{k_0}{T} \left(s_1 \sin s_1\theta + \frac{1}{T} \cos s_1\theta + s_2 \sinh s_2\theta - \frac{1}{T} \cosh s_2\theta \right) \tag{2.101}$$

$$r_{12}(\theta) = \frac{1}{T_i} \left(\frac{1}{T} \cos s_1\theta + s_1 \sin s_1\theta - \frac{1}{T} \cosh s_2\theta + s_2 \sinh s_2\theta \right) \tag{2.102}$$

$$r_{22}(\theta) = s_2^2 \cos s_1\theta + s_1^2 \cosh s_2\theta \tag{2.103}$$

$$r_{32}(\theta) = \frac{k_0}{T T_i^2} \left(\frac{1}{s_1} \sin s_1\theta - \frac{1}{s_2} \sinh s_2\theta \right) \tag{2.104}$$

$$r_{42}(\theta) = \frac{k_0}{T T_i} \left(-\cos s_1\theta + \frac{1}{T s_1} \sin s_1\theta - \frac{1}{T s_2} \sinh s_2\theta + \cosh s_2\theta \right) \tag{2.105}$$

$$r_{13}(\theta) = \frac{k_0}{T T_i} \Big(\cos s_1\theta - \cosh s_2\theta \Big) \tag{2.106}$$

$$r_{23}(\theta) = \frac{k_0}{T} \left(-s_1 \sin s_1\theta + \frac{1}{T} \cos s_1\theta - s_2 \sinh s_2\theta - \frac{1}{T} \cosh s_2\theta \right) \tag{2.107}$$

$$r_{33}(\theta) = s_1^2 \cos s_1\theta + \frac{s_1}{T} \sin s_1\theta + s_2^2 \cosh s_2\theta + \frac{s_2}{T} \sinh s_2\theta \tag{2.108}$$

$$r_{43}(\theta) = \frac{k_0^2}{T^2 T_i} \left(\frac{1}{s_1} \sin s_1\theta - \frac{1}{s_2} \sinh s_2\theta \right) \tag{2.109}$$

$$r_{14}(\theta) = \frac{k_0}{T T_i^2} \left(-\frac{1}{s_1} \sin s_1\theta + \frac{1}{s_2} \sinh s_2\theta \right) \tag{2.110}$$

$$r_{24}(\theta) = \frac{k_0}{T T_i} \left(-\cos s_1\theta - \frac{1}{T s_1} \sin s_1\theta + \frac{1}{T s_2} \sinh s_2\theta + \cosh s_2\theta \right) \tag{2.111}$$

$$r_{34}(\theta) = \frac{1}{T_i} \left(\frac{1}{T} \cos s_1\theta - s_1 \sin s_1\theta - \frac{1}{T} \cosh s_2\theta - s_2 \sinh s_2\theta \right) \tag{2.112}$$

$$r_{44}(\theta) = s_2^2 \cos s_1\theta + s_1^2 \cosh s_2\theta \tag{2.113}$$

The solution of the differential equations (2.95) is given by the terms

$$\beta_{12}(\theta) = \frac{1}{s_1^2+s_2^2}\Big[r_{11}(\theta+r)\beta_{12}(-r)+r_{12}(\theta+r)\beta_{22}(-r)+ \\ +r_{13}(\theta+r)\kappa_{12}(-r)-r_{14}(\theta+r)\Big] \tag{2.114}$$

$$\beta_{22}(\theta) = \frac{1}{s_1^2+s_2^2}\Big[r_{21}(\theta+r)\beta_{12}(-r)+r_{22}(\theta+r)\beta_{22}(-r)+ \\ +r_{23}(\theta+r)\kappa_{12}(-r)-r_{24}(\theta+r)\Big] \tag{2.115}$$

$$\kappa_{12}(\theta) = \frac{1}{s_1^2+s_2^2}\Big[r_{31}(\theta+r)\beta_{12}(-r)+r_{32}(\theta+r)\beta_{22}(-r)+ \\ +r_{33}(\theta+r)\kappa_{12}(-r)-r_{34}(\theta+r)\Big] \tag{2.116}$$

$$\kappa_{22}(\theta) = \frac{1}{s_1^2+s_2^2}\Big[r_{41}(\theta+r)\beta_{12}(-r)+r_{42}(\theta+r)\beta_{22}(-r)+ \\ +r_{43}(\theta+r)\kappa_{12}(-r)-r_{44}(\theta+r)\Big] \tag{2.117}$$

The matrix α and the initial conditions $\beta_{12}(-r)$, $\beta_{22}(-r)$, $\kappa_{12}(-r)$ are obtained from the set of algebraic equations

$$\begin{cases} -\frac{2}{T}\alpha_{11}+\frac{2}{T_i}\alpha_{12} = -1 \\ -\frac{2}{T}\alpha_{12}+\frac{2}{T_i}\alpha_{22}+\kappa_{12}(-r) = 0 \\ -\frac{2k_0}{T}\alpha_{11}-\beta_{12}(-r) = 0 \\ -\frac{2k_0}{T}\alpha_{12}-\beta_{22}(-r) = 0 \\ q_{11}\beta_{12}(-r)+q_{12}\beta_{22}(-r)+q_{13}\kappa_{12}(-r) = q_{14} \\ q_{21}\beta_{12}(-r)+q_{22}\beta_{22}(-r)+q_{23}\kappa_{12}(-r) = q_{24} \end{cases} \tag{2.118}$$

where

$$q_{11} = \left(s_1^2-\frac{k_0}{TT_i}\right)\cos\frac{s_1 r}{2}-\frac{s_1}{T}\sin\frac{s_1 r}{2}+\left(s_2^2+\frac{k_0}{TT_i}\right)\cosh\frac{s_2 r}{2}-\frac{s_2}{T}\sinh\frac{s_2 r}{2} \tag{2.119}$$

$$q_{12} = \frac{1}{TT_i}\cos\frac{s_1 r}{2}+\frac{s_1^2-\frac{k_0}{TT_i}}{T_i s_1}\sin\frac{s_1 r}{2}-\frac{1}{TT_i}\cosh\frac{s_2 r}{2}+\frac{s_2^2+\frac{k_0}{TT_i}}{T_i s_2}\sinh\frac{s_2 r}{2} \tag{2.120}$$

$$q_{13} = \left(\frac{k_0}{TT_i} - s_1^2\right)\cos\frac{s_1 r}{2} - \frac{s_1}{T}\sin\frac{s_1 r}{2} - \left(\frac{k_0}{TT_i} + s_2^2\right)\cosh\frac{s_2 r}{2} - \frac{s_2}{T}\sinh\frac{s_2 r}{2} \tag{2.121}$$

$$q_{14} = -\frac{1}{TT_i}\cos\frac{s_1 r}{2} + \frac{s_1^2 - \frac{k_0}{TT_i}}{T_i s_1}\sin\frac{s_1 r}{2} + \frac{1}{TT_i}\cosh\frac{s_2 r}{2} + \frac{s_2^2 + \frac{k_0}{TT_i}}{T_i s_2}\sinh\frac{s_2 r}{2} \tag{2.122}$$

$$q_{21} = -\frac{k_0}{T^2}\cos\frac{s_1 r}{2} - \frac{k_0\left(s_1^2 + \frac{k_0}{TT_i}\right)}{Ts_1}\sin\frac{s_1 r}{2} + \frac{k_0}{T^2}\cosh\frac{s_2 r}{2} - \frac{k_0\left(s_2^2 - \frac{k_0}{TT_i}\right)}{Ts_2}\sinh\frac{s_2 r}{2} \tag{2.123}$$

$$q_{22} = \left(s_2^2 + \frac{k_0}{TT_i}\right)\cos\frac{s_1 r}{2} - \frac{k_0}{T^2 T_i s_1}\sin\frac{s_1 r}{2} + \left(s_1^2 - \frac{k_0}{TT_i}\right)\cosh\frac{s_2 r}{2} + \frac{k_0}{T^2 T_i s_2}\sinh\frac{s_2 r}{2} \tag{2.124}$$

$$q_{23} = \frac{k_0}{T^2}\cos\frac{s_1 r}{2} - \frac{k_0\left(s_1^2 + \frac{k_0}{TT_i}\right)}{Ts_1}\sin\frac{s_1 r}{2} - \frac{k_0}{T^2}\cosh\frac{s_2 r}{2} - \frac{k_0\left(s_2^2 - \frac{k_0}{TT_i}\right)}{Ts_2}\sinh\frac{s_2 r}{2} \tag{2.125}$$

$$\begin{aligned} q_{24} = &-\left(s_2^2 + \frac{k_0}{TT_i}\right)\cos\frac{s_1 r}{2} - \frac{k_0}{T^2 T_i s_1}\sin\frac{s_1 r}{2} + \\ &+\left(-s_1^2 + \frac{k_0}{TT_i}\right)\cosh\frac{s_2 r}{2} + \frac{k_0}{T^2 T_i s_2}\sinh\frac{s_2 r}{2} \end{aligned} \tag{2.126}$$

The solution of the set of equations (2.118) has a form

$$\begin{aligned} \alpha_{11} = \frac{1}{M}\Bigg[&\left(k_0 + \frac{T}{T_i}\right)\left(s_1^2 + s_2^2\right)\cos\frac{s_1 r}{2}\cosh\frac{s_2 r}{2} + \frac{k_0(T - k_0 T_i)(s_1^2 + s_2^2)}{T_i^2 T s_1 s_2}\sin\frac{s_1 r}{2}\sinh\frac{s_2 r}{2} + \\ &+\frac{\left(s_2^2 - \frac{k_0}{TT_i}\right)\left[1 + 2k_0^2 + k_0 T T_i(s_1^2 + s_2^2)\right]}{T_i s_2}\cos\frac{s_1 r}{2}\sinh\frac{s_2 r}{2} + \\ &+\frac{\left(s_1^2 + \frac{k_0}{TT_i}\right)\left[1 + 2k_0^2 - k_0 T T_i(s_1^2 + s_2^2)\right]}{T_i s_1}\sin\frac{s_1 r}{2}\cosh\frac{s_2 r}{2}\Bigg] \end{aligned} \tag{2.127}$$

$$\begin{aligned} \alpha_{12} = \frac{1}{M}\Bigg[&\left(s_1^2 + s_2^2\right)\cos\frac{s_1 r}{2}\cosh\frac{s_2 r}{2} + \frac{k_0(s_1^2 + s_2^2)}{TT_i s_1 s_2}\sin\frac{s_1 r}{2}\sinh\frac{s_2 r}{2} + \\ &+\frac{\left(s_2^2 - \frac{k_0}{TT_i}\right)\left(1 + 2k_0^2\right)}{Ts_2}\cos\frac{s_1 r}{2}\sinh\frac{s_2 r}{2} + \frac{\left(s_1^2 + \frac{k_0}{TT_i}\right)\left(1 + 2k_0^2\right)}{Ts_1}\sin\frac{s_1 r}{2}\cosh\frac{s_2 r}{2}\Bigg] \end{aligned} \tag{2.128}$$

$$\alpha_{22} = \frac{1}{M}\Bigg[\frac{(k_0T + k_0^2T_i + T_i)(s_1^2+s_2^2)}{T}\cos\frac{s_1r}{2}\cosh\frac{s_2r}{2} + $$
$$+\frac{T_is_2^2\left(s_2^2+\frac{k_0^2}{T^2}\right)+\frac{k_0}{T}\left(s_1^2-\frac{k_0^2}{T^2}\right)}{s_2}\cos\frac{s_1r}{2}\sinh\frac{s_2r}{2}+$$
$$+\frac{T_is_1^2\left(-s_1^2+\frac{k_0^2}{T^2}\right)+\frac{k_0}{T}\left(s_2^2+\frac{k_0^2}{T^2}\right)}{s_1}\sin\frac{s_1r}{2}\cosh\frac{s_2r}{2}+$$
$$+\frac{k_0(-k_0T+k_0^2T_i+T_i)(s_1^2+s_2^2)}{T^2T_is_1s_2}\sin\frac{s_1r}{2}\sinh\frac{s_2r}{2}\Bigg] \quad (2.129)$$

$$\beta_{12}(-r) = -\frac{2k_0}{TM}\Bigg[\left(k_0+\frac{T}{T_i}\right)\left(s_1^2+s_2^2\right)\cos\frac{s_1r}{2}\cosh\frac{s_2r}{2}+$$
$$+\frac{\left(s_2^2-\frac{k_0}{TT_i}\right)\left[1+2k_0^2+k_0TT_i(s_1^2+s_2^2)\right]}{T_is_2}\cos\frac{s_1r}{2}\sinh\frac{s_2r}{2}+$$
$$+\frac{\left(s_1^2+\frac{k_0}{TT_i}\right)\left[1+2k_0^2-k_0TT_i(s_1^2+s_2^2)\right]}{T_is_1}\sin\frac{s_1r}{2}\cosh\frac{s_2r}{2}+$$
$$+\frac{k_0(T-k_0T_i)(s_1^2+s_2^2)}{T_i^2Ts_1s_2}\sin\frac{s_1r}{2}\sinh\frac{s_2r}{2}\Bigg] \quad (2.130)$$

$$\beta_{22}(-r) = -\frac{2k_0}{TM}\Bigg[(s_1^2+s_2^2)\cos\frac{s_1r}{2}\cosh\frac{s_2r}{2}+\frac{k_0(s_1^2+s_2^2)}{TT_is_1s_2}\sin\frac{s_1r}{2}\sinh\frac{s_2r}{2}+$$
$$+\frac{\left(s_2^2-\frac{k_0}{TT_i}\right)\left(1+2k_0^2\right)}{Ts_2}\cos\frac{s_1r}{2}\sinh\frac{s_2r}{2}+\frac{\left(s_1^2+\frac{k_0}{TT_i}\right)\left(1+2k_0^2\right)}{Ts_1}\sin\frac{s_1r}{2}\cosh\frac{s_2r}{2}\Bigg] \quad (2.131)$$

$$\kappa_{12}(-r) = \frac{2}{TM}\Bigg[-\frac{k_0(k_0T_i+T)(s_1^2+s_2^2)}{T_i}\cos\frac{s_1r}{2}\cosh\frac{s_2r}{2}+$$
$$+\left[(k_0T_i+T)s_1s_2^2-\frac{k_0s_1}{TT_i}(T-k_0T_i)\right]\sin\frac{s_1r}{2}\cosh\frac{s_2r}{2}+$$
$$+\left[\frac{k_0s_2(k_0T_i-T)}{TT_i}-(k_0T_i+T)s_1^2s_2\right]\cos\frac{s_1r}{2}\sinh\frac{s_2r}{2}+$$
$$+T(T-k_0T_i)(s_1^2+s_2^2)s_1s_2\sin\frac{s_1r}{2}\sinh\frac{s_2r}{2}\Bigg] \quad (2.132)$$

where

$$M=-\frac{2k_0}{T}\left(s_1^2+s_2^2\right)\left[-\cos\frac{s_1r}{2}\cosh\frac{s_2r}{2}+\frac{k_0}{TT_is_1s_2}\sin\frac{s_1r}{2}\sinh\frac{s_2r}{2}+\right.$$

$$\left.+\left(\frac{k_0}{T_is_2}-Ts_2\right)\cos\frac{s_1r}{2}\sinh\frac{s_2r}{2}+\left(\frac{k_0}{T_is_1}+Ts_1\right)\sin\frac{s_1r}{2}\cosh\frac{s_2r}{2}\right] \tag{2.133}$$

According to the formula (2.89) the value of index is given by a term

$$J=V(y(0),y(\theta)) \tag{2.134}$$

After calculations one obtains

$$J=x_0^2\alpha_{11}-2x_0z_0\alpha_{12}+z_0^2\alpha_{22}-\frac{x_0z_0}{s_1^2+s_2^2}\left[\left(s_1\sin s_1r+\frac{1}{T}\cos s_1r+s_2\sinh s_2r+\right.\right.$$

$$\left.-\frac{1}{T}\cosh s_2r\right)\beta_{12}(-r)+\frac{1}{TT_i}\left(\frac{1}{s_1}\sin s_1r-\frac{1}{s_2}\sinh s_2r-T\cos s_1r+T\cosh s_2r\right)\beta_{22}(-r)+$$

$$\left.+\frac{k_0}{TT_i}\left(\frac{1}{s_1}\sin s_1r-\frac{1}{s_2}\sinh s_2r\right)\kappa_{12}(-r)-\frac{k_0}{TT_i^2}\left(\frac{1}{s_1^2}\cos s_1r+\frac{1}{s_2^2}\cosh s_2r-\frac{1}{s_1^2}-\frac{1}{s_2^2}\right)\right]+$$

$$+\frac{z_0^2k_0r}{T^2T_i(s_1^2+s_2^2)}\left[k_0\left(\frac{1}{s_2}\sinh s_2r-\frac{1}{s_1}\sin s_1r\right)\beta_{12}(-r)+\frac{k_0}{T_i}\left(\frac{1}{s_1^2}\cos s_1r+\right.\right.$$

$$\left.+\frac{1}{s_2^2}\cosh s_2r\right)\beta_{22}(-r)+T_i(\cos s_1r-Ts_1\sin s_1r-\cosh s_2r-Ts_2\sinh s_2r)\kappa_{12}(-r)+$$

$$\left.+\frac{1}{s_1}\sin s_1r+T\cos s_1r-\frac{1}{s_2}\sinh s_2r-T\cosh s_2r\right] \tag{2.135}$$

Figure 2.3 shows the value of the index $J(1/T_i)$ for $x_0=1$, $z_0=1$, $k_0=1$, $T=5$ and $r=1$. You can see that there exists a critical value of the $1/T_{icrit}$. The system (2.80) is stable for $1/T_i$ less then critical one and unstable for $1/T_i$ greater then critical.
Figure 2.4 shows the value of the index $J(1/T_i)$ for $1/T_i$ less then critical one. You can see that the function $J(1/T_i)$ is convex and has a minimum.
We search for an optimal time of isodrome which minimizes the index (2.135). Optimization results, obtained by means of the Matlab function *fminsearch*, are given in Table 2.2. These results are obtained for $x_0=1$, $z_0=1$, $k_0=1$, and $T=5$. Critical time of isodrome is a maximal admissible time of isodrome for system (2.77). System (2.77) is unstable for time of isodrome less then critical one.

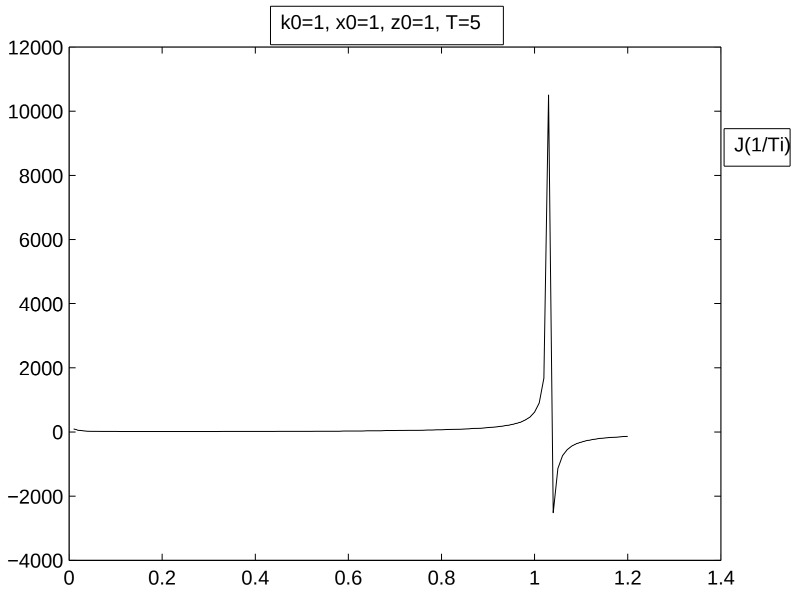

Fig. 2.3. Value of the index $J(1/T_i)$ for $1/T_i$ greater then critical one

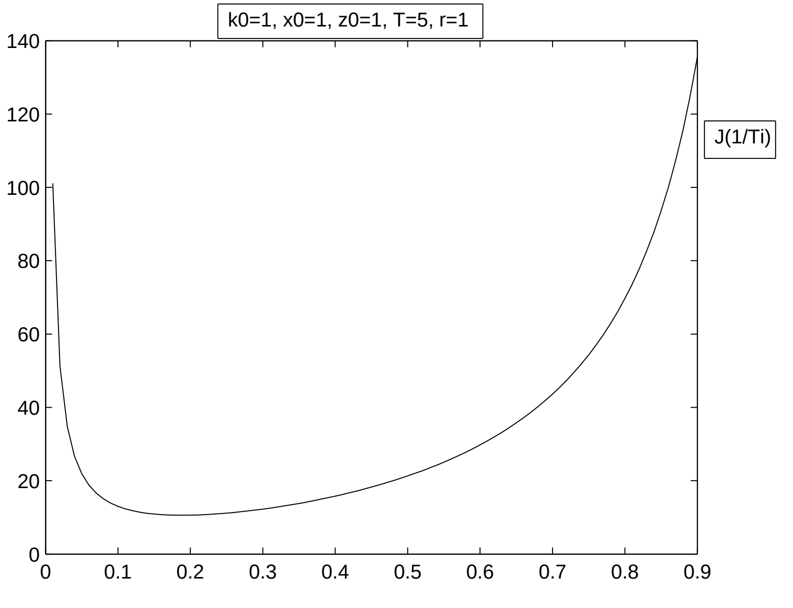

Fig. 2.4. Value of the index $J(1/T_i)$ for $1/T_i$ less then critical one

Table 2.2
Optimization results

Delay r	Optimal $1/T_i$	Index value	Critical $1/T_i$
1.0	0.1879	10.5870	1.03
1.5	0.1602	12.3667	0.69
2.0	0.1396	14.1349	0.53
2.5	0.1237	15.8853	0.42
3.0	0.1112	17.6164	0.36
3.5	0.1010	19.3293	0.31
4.0	0.0926	21.0264	0.27
4.5	0.0856	22.7108	0.24
5.0	0.0796	4.3854	0.22

2.3 The Lyapunov functional for a linear system with two delays

2.3.1 Mathematical model of a linear time delay system with two delays

Let us consider a linear system with two delays, whose dynamics is described by equation [12]

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx(t-r_2) + Cx(t-r_1) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{2.136}$$

$t \geq t_0,\ \theta \in [-r_2, 0),\ r_2 \geq r_1 \geq 0,\ A, B, C \in \mathbb{R}^{n\times n},\ x_0 \in \mathbb{R}^n,\ \varphi \in L^2([-r_2,0),\mathbb{R}^n)$, where $L^2([-r_2,0),\mathbb{R}^n)$ is a space of Lebesgue square integrable functions on interval $[-r_2,0)$ with values in $\mathbb{R}^n$.

The solution of the functional-differential equation (2.136) with initial value (x_0,φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$ and is denoted as $x(\cdot,t_0,(x_0,\varphi))$.

The function x_t is a shifted restriction of the function $x(\cdot,t_0,(x_0,\varphi))$ to the interval $[t-r_2,t)$.

The state is a vector $S = \begin{bmatrix} x(t) \\ x_t \end{bmatrix}$. The state space is given by the Cartesian product

$$X = \mathbb{R}^n \times L^2([-r_2,0),\mathbb{R}^n) \tag{2.137}$$

2.3.2 Determination of the Lyapunov functional

On the state space X we define a quadratic functional V, positive definite, differentiable, given by the formula [12]

$$V(x(t),x_t) = x^T(t)\alpha x(t) + \int_{-r_2}^{0} x^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r_1}^{0} x^T(t)\kappa(\sigma)x_t(\sigma)d\sigma +$$
$$+\int_{-r_2}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta_1(\theta,\xi)x_t(\xi)d\xi d\theta + \int_{-r_1}^{0}\int_{\sigma}^{0} x_t^T(\sigma)\delta_2(\sigma,\varsigma)x_t(\varsigma)d\varsigma d\sigma +$$
$$+\int_{-r_2}^{0}\int_{-r_1}^{0} x_t^T(\theta)\delta_3(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{2.138}$$

for $t \geq t_0$, where $\alpha = \alpha^T \in \mathbb{R}^{n\times n}$, $\beta \in C^1([-r_2,0],\mathbb{R}^{n\times n})$, $\kappa \in C^1([-r_1,0],\mathbb{R}^{n\times n})$,
$\delta_1 \in C^1(\Omega_1,\mathbb{R}^{n\times n})$, $\delta_2 \in C^1(\Omega_2,\mathbb{R}^{n\times n})$, $\delta_3 \in C^1(\Omega_3,\mathbb{R}^{n\times n})$,
$\Omega_1 = \{(\theta,\xi):\ \theta \in [-r_2,0],\ \xi \in [\theta,0]\}$,
$\Omega_2 = \{(\sigma,\varsigma):\ \sigma \in [-r_1,0],\ \varsigma \in [\sigma,0]\}$, $\Omega_3 = \{(\theta,\sigma):\ \theta \in [-r_2,0],\ \sigma \in [-r_1,0]\}$
C^1 is a space of continuous functions with continuous derivative.
It is taken the following procedure of determination of the functional (2.138) coefficients. One computes the time derivative of each term of the right-hand-side of the formula (2.138) and one substitutes in place of $dx(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dx(t)}{dt} = Ax(t) + Bx_t(-r_2) + Cx_t(-r_1) \tag{2.139}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{2.140}$$

In such a manner one attains

$$\frac{dV(x(t),x_t)}{dt} = x^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} + \frac{\kappa(0)+\kappa^T(0)}{2}\right]x(t) +$$
$$+x^T(t)\left[2\alpha B - \beta(-r_2)\right]x_t(-r_2) + x^T(t)\left[2\alpha C - \kappa(-r_1)\right]x_t(-r_1) +$$
$$+\int_{-r_2}^{0} x^T(t)\left[A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta_1^T(\theta,0) + \delta_3^T(\theta,0)\right]x_t(\theta)d\theta +$$
$$+\int_{-r_2}^{0} x_t^T(-r_2)\left[B^T\beta(\theta) - \delta_1(-r_2,\theta)\right]x_t(\theta)d\theta +$$
$$+\int_{-r_2}^{0} x_t^T(-r_1)\left[C^T\beta(\theta) - \delta_3^T(\theta,-r_1)\right]x_t(\theta)d\theta +$$

$$+\int_{-r_1}^{0} x^T(t)\left[A^T\kappa(\sigma)-\frac{d\kappa(\sigma)}{d\sigma}+\delta_2^T(\sigma,0)+\delta_3(0,\sigma)\right]x_t(\sigma)d\sigma+$$

$$+\int_{-r_1}^{0} x_t^T(-r_1)\left[C^T\kappa(\sigma)-\delta_2(-r_1,\sigma)\right]x_t(\sigma)d\sigma+$$

$$+\int_{-r_1}^{0} x_t^T(-r_2)\left[B^T\kappa(\sigma)-\delta_3(-r_2,\sigma)\right]x_t(\sigma)d\sigma+$$

$$-\int_{-r_2}^{0}\int_{\theta}^{0} x_t^T(\theta)\left[\frac{\partial\delta_1(\theta,\xi)}{\partial\theta}+\frac{\partial\delta_1(\theta,\xi)}{\partial\xi}\right]x_t(\xi)d\xi d\theta+$$

$$-\int_{-r_1}^{0}\int_{\sigma}^{0} x_t^T(\sigma)\left[\frac{\partial\delta_2(\sigma,\varsigma)}{\partial\sigma}+\frac{\partial\delta_2(\sigma,\varsigma)}{\partial\varsigma}\right]x_t(\varsigma)d\varsigma d\sigma+$$

$$-\int_{-r_2}^{0}\int_{-r_1}^{0} x_t^T(\theta)\left[\frac{\partial\delta_3(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta_3(\theta,\sigma)}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta \tag{2.141}$$

To achieve negative definiteness of that derivative we assume that

$$\frac{dV(x(t),x_t)}{dt}\equiv -x^T(t)x(t) \tag{2.142}$$

From equations (2.141) and (2.142) one obtains a system of equations

$$A^T\alpha+\alpha A+\frac{\beta(0)+\beta^T(0)}{2}+\frac{\kappa(0)+\kappa^T(0)}{2}=-I \tag{2.143}$$

$$2\alpha B-\beta(-r_2)=0 \tag{2.144}$$

$$2\alpha C-\kappa(-r_1)=0 \tag{2.145}$$

$$A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta_1^T(\theta,0)+\delta_3^T(\theta,0)=0 \tag{2.146}$$

$$B^T\beta(\theta)-\delta_1(-r_2,\theta)=0 \tag{2.147}$$

$$C^T\beta(\theta)-\delta_3^T(\theta,-r_1)=0 \tag{2.148}$$

$$A^T\kappa(\sigma)-\frac{d\kappa(\sigma)}{d\sigma}+\delta_2^T(\sigma,0)+\delta_3(0,\sigma)=0 \tag{2.149}$$

$$C^T \kappa(\sigma) - \delta_2(-r_1, \sigma) = 0 \tag{2.150}$$

$$B^T \kappa(\sigma) - \delta_3(-r_2, \sigma) = 0 \tag{2.151}$$

$$\frac{\partial \delta_1(\theta, \xi)}{\partial \theta} + \frac{\partial \delta_1(\theta, \xi)}{\partial \xi} = 0 \tag{2.152}$$

$$\frac{\partial \delta_2(\sigma, \varsigma)}{\partial \sigma} + \frac{\partial \delta_2(\sigma, \varsigma)}{\partial \varsigma} = 0 \tag{2.153}$$

$$\frac{\partial \delta_3(\theta, \sigma)}{\partial \theta} + \frac{\partial \delta_3(\theta, \sigma)}{\partial \sigma} = 0 \tag{2.154}$$

for $\theta \in [-r_2, 0]$, $\sigma \in [-r_1, 0]$, $\xi \in [\theta, 0]$, $\varsigma \in [\sigma, 0]$.
The solutions of equations (2.152)-(2.154) are functions

$$\delta_i(\theta, \sigma) = f_i(\theta - \sigma) \tag{2.155}$$

where $f_i \in C^1([-r_2, r_1])$ for $i = 1, 2, 3$.
From equations (2.147) and (2.155) we obtain

$$\delta_1(-r_2, \theta) = f_1(-\theta - r_2) = B^T \beta(\theta) \tag{2.156}$$

Hence

$$\delta_1^T(\theta, 0) = f_1^T(\theta) = \beta^T(-\theta - r_2) B \tag{2.157}$$

From equations (2.148) and (2.155) we obtain

$$\delta_3^T(\theta, -r_1) = f_3^T(\theta + r_1) = C^T \beta(\theta) \tag{2.158}$$

Hence

$$\delta_3^T(\theta, 0) = f_3^T(\theta) = C^T \beta(\theta - r_1) \tag{2.159}$$

When we put (2.157) and (2.159) into (2.146), we get the formula

$$\frac{d\beta(\theta)}{d\theta} = A^T \beta(\theta) + \beta^T(-\theta - r_2) B + C^T \beta(\theta - r_1) \tag{2.160}$$

for $\theta \in [-r_2, 0]$.
From equation (2.150) we obtain

$$\delta_2(-r_1, \sigma) = f_2(-\sigma - r_1) = C^T \kappa(\sigma) \tag{2.161}$$

Hence

$$\delta_2^T(\sigma, 0) = f_2^T(\sigma) = \kappa^T(-\sigma - r_1) C \tag{2.162}$$

From equation (2.151) we obtain

$$\delta_3(-r_2,\sigma) = f_3(-\sigma - r_2) = B^T \kappa(\sigma) \tag{2.163}$$

Hence

$$\delta_3(0,\sigma) = f_3(-\sigma) = B^T \kappa(\sigma - r_2) \tag{2.164}$$

When we put (2.30) and (2.164) into (2.149), we get the formula

$$\frac{d\kappa(\sigma)}{d\sigma} = A^T \kappa(\sigma) + \kappa^T(-\sigma - r_1)C + B^T \kappa(\sigma - r_2) \tag{2.165}$$

for $\sigma \in [-r_1, 0]$.
We introduce two new functions

$$\eta(\theta) = \beta(-\theta - r_2) \tag{2.166}$$

$$\vartheta(\sigma) = \kappa(-\sigma - r_1) \tag{2.167}$$

for $\theta \in [-r_2, 0]$, $\sigma \in [-r_1, 0]$.
We calculate the derivatives of (2.34) and (2.167)

$$\frac{d\eta(\theta)}{d\theta} = -\beta^T(\theta)B - A^T \eta(\theta) - C^T \eta(\theta + r_1) \tag{2.168}$$

$$\frac{d\vartheta(\sigma)}{d\sigma} = -\kappa^T(\sigma)C - A^T \vartheta(\sigma) - B^T \vartheta(\sigma + r_2) \tag{2.169}$$

for $\theta \in [-r_2, 0]$, $\sigma \in [-r_1, 0]$.
We obtained the system of differential equations

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} = A^T \beta(\theta) + \eta^T(\theta)B + C^T \beta(\theta - r_1) \\ \dfrac{d\eta(\theta)}{d\theta} = -\beta^T(\theta)B - A^T \eta(\theta) - C^T \eta(\theta + r_1) \\ \dfrac{d\kappa(\sigma)}{d\sigma} = A^T \kappa(\sigma) + \vartheta^T(\sigma)C + B^T \kappa(\sigma - r_2) \\ \dfrac{d\vartheta(\sigma)}{d\sigma} = -\kappa^T(\sigma)C - A^T \vartheta(\sigma) - B^T \vartheta(\sigma + r_2) \end{cases} \tag{2.170}$$

for $\theta \in [-r_2, 0]$, $\sigma \in [-r_1, 0]$.
Relations (2.34) and (2.167) implies $\beta(-r_2) = \eta(0)$ and $\kappa(-r_1) = \vartheta(0)$ and

$$\beta(\theta) \mid_{\theta = -\frac{r_2}{2}} = \eta(\theta) \mid_{\theta = -\frac{r_2}{2}} \tag{2.171}$$

$$\kappa(\sigma) \mid_{\sigma = -\frac{r_1}{2}} = \vartheta(\sigma) \mid_{\sigma = -\frac{r_1}{2}} \tag{2.172}$$

Equations (2.143), (2.144) and (2.145) take a form

$$A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} + \frac{\kappa(0)+\kappa^T(0)}{2} = -I \tag{2.173}$$

$$2\alpha B - \eta(0) = 0 \tag{2.174}$$

$$2\alpha C - \vartheta(0) = 0 \tag{2.175}$$

The set of algebraic equations (2.171)–(2.175) enables determination of the initial conditions of the differential equations (2.170) and the matrix α.
Matrix $\delta_1(\theta,\sigma)$ we obtain from equations (2.155), (2.157) and (2.34)

$$\delta_1(\theta,\sigma) = B^T\eta(\theta-\sigma) \tag{2.176}$$

Matrix $\delta_2(\theta,\sigma)$ we obtain from equations (2.155), (2.30), (2.167)

$$\delta_2(\theta,\sigma) = C^T\vartheta(\theta-\sigma) \tag{2.177}$$

Matrix $\delta_3(\theta,\sigma)$ we obtain from equations (2.155), (2.159), (2.164)

$$\delta_3(\theta,\sigma) = B^T\kappa(\sigma-\theta-r_2) \tag{2.178}$$

In this way we obtained all parameters of the functional (3.6).

2.3.3 Solution of the set of differential equations (2.170) for commensurate delays

Functions β, η, κ, ϑ are not independent, β and η are linked by formula (2.34), κ and ϑ by formula (2.167). The functions β and κ are also combined. This is implied by formulas (2.159) and (2.163). From (2.159) we obtain

$$f_3(\theta) = \beta^T(\theta-r_1)C \tag{2.179}$$

and from (2.163) we have

$$f_3(\sigma) = B^T\kappa(-\sigma-r_2) \tag{2.180}$$

According to (2.155), function f_3 is defined on the interval $[-r_2, r_1]$. Now we can write down the following functional interdependences between the functions β, η, κ, ϑ

$$C^T\beta(\theta-r_1) = \vartheta^T(\theta+r_2-r_1)B \quad for\ \theta\in[-r_2,-r_2+r_1] \tag{2.181}$$

$$C^T\eta(\theta+r_1) = \kappa^T(\theta)B \quad for\ \theta\in[-r_1,0] \tag{2.182}$$

$$B^T\kappa(\sigma-r_2) = \eta^T(\sigma-r_2+r_1)C \quad for\ \sigma\in[-r_1,-r_1+r_2-r_1] \tag{2.183}$$

$$B^T\vartheta(\sigma+r_2) = \beta^T(\sigma)C \quad for\ \sigma\in[-r_1,0] \tag{2.184}$$

Let us consider a special case, in which the system of equations (2.170) will be transformed into the set of ordinary differential equations.
We assume that the following relationships hold

$$r_1 = mh;\ r_2 = nh;\ m,n \in \mathbb{N};\ n \geq m;\ \mathbb{R} \ni h > 0 \tag{2.185}$$

We introduce the functions

$$\beta_i(\xi),\ \eta_i(\xi),\ \kappa_j(\xi),\ \vartheta_j(\xi)$$

for $\xi \in [-h,0];\ i = 1,2,...,n;\ j = 1,2,...,m$ defined by formulas

$$\beta_i(\theta) = \beta(\theta)\ for\ \theta \in [-r_2 + (i-1)h, -r_2 + ih],\ i = 1,...,n \tag{2.186}$$

$$\eta_i(\theta) = \eta(\theta)\ for\ \theta \in [-r_2 + (i-1)h, -r_2 + ih],\ i = 1,...,n \tag{2.187}$$

$$\kappa_j(\sigma) = \kappa(\sigma)\ for\ \sigma \in [-r_1 + (j-1)h, -r_1 + jh],\ j = 1,...,m \tag{2.188}$$

$$\vartheta_j(\sigma) = \vartheta(\sigma)\ for\ \sigma \in [-r_1 + (j-1)h, -r_1 + jh],\ j = 1,...,m \tag{2.189}$$

These functions satisfy the following set of conditions

$$\begin{cases} \beta_1(-h) = \beta(-r_2) = \eta(0) \\ \beta_i(-h) = \beta_{i-1}(0) \quad for\ i = 2,...,n \\ \eta_1(-h) = \eta(-r_2) = \beta(0) \\ \eta_i(-h) = \eta_{i-1}(0) \quad for\ i = 2,...,n \\ \kappa_1(-h) = \kappa(-r_1) = \vartheta(0) \\ \kappa_j(-h) = \kappa_{j-1}(0) \quad for\ j = 2,...,m \\ \vartheta_1(-h) = \vartheta(-r_1) = \kappa(0) \\ \vartheta_j(-h) = \vartheta_{j-1}(0) \quad for\ j = 2,...,m \end{cases} \tag{2.190}$$

We can write equations (2.170) with regard to dependencies (2.181)–(2.184) for functions (2.186)–(2.189) in a form

$$\begin{cases} \dfrac{d\beta_i(\xi)}{d\xi} = A^T\beta_i(\xi) + \eta_i^T(\xi)B + \vartheta_i^T(\xi)B \; for \; i = 1,...,m \\ \dfrac{d\beta_i(\xi)}{d\xi} = A^T\beta_i(\xi) + \eta_i^T(\xi)B + C^T\beta_{i-m}(\xi) \; for \; i = m+1,...,n \\ \dfrac{d\eta_i(\xi)}{d\xi} = -\beta_i^T(\xi)B - A^T\eta_i(\xi) - C^T\eta_{i+m}(\xi) \; for \; i = 1,...,n-m \\ \dfrac{d\eta_i(\xi)}{d\xi} = -\beta_i^T(\xi)B - A^T\eta_i(\xi) - \kappa_{i-(n-m)}^T(\xi)B \; for \; i = n-m+1,...,n \\ \dfrac{d\kappa_j(\xi)}{d\xi} = A^T\kappa_j(\xi) + \vartheta_j^T(\xi)C + \eta_j^T(\xi)C \; for \; j = 1,...,n-m \\ \dfrac{d\kappa_j(\xi)}{d\xi} = A^T\kappa_j(\xi) + \vartheta_j^T(\xi)C + B^T\kappa_{j-(n-m)}(\xi) \; for \; j = n-m+1,...,m \\ \dfrac{d\vartheta_j(\xi)}{d\xi} = -\kappa_j^T(\xi)C - A^T\vartheta_j(\xi) - \beta_{j+n-m}^T(\xi)C \; for \; j = 1,...,m \end{cases} \tag{2.191}$$

for $\xi \in [-h, 0]$.
There are relationships between the initial conditions of system (2.191) as below

$$\begin{cases} \beta_i(0) = \eta_{n-i}(0) \quad for \; i = 1,...,n-1 \\ \beta_n(0) = \beta(0) \\ \vartheta_j(0) = \kappa_{m-j}(0) \quad for \; j = 1,...,m-1 \\ \vartheta_m(0) = \vartheta(0) \end{cases} \tag{2.192}$$

We obtain matrix α and the initial conditions of the system (2.191) by solving the set of algebraic equations

$$\begin{cases} A^T\alpha + \alpha A + \dfrac{\beta(0) + \beta^T(0)}{2} + \dfrac{\kappa(0) + \kappa^T(0)}{2} = -I \\ 2\alpha B - \eta(0) = 0 \\ 2\alpha C - \vartheta(0) = 0 \\ \beta_i(\xi)\,|_{\xi=-\frac{h}{2}} = \eta_{n+1-i}(\xi)\,|_{\xi=-\frac{h}{2}} \quad for \; i = 1,...,n \\ \kappa_j(\xi)\,|_{\xi=-\frac{h}{2}} = \vartheta_{m+1-j}(\xi)\,|_{\xi=-\frac{h}{2}} \quad for \; j = 1,...,m \end{cases} \tag{2.193}$$

Having a solution of the set of equations (2.191) we can obtain the matrices $\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$ from equations (2.186)–(2.189) and the matrices $\delta_1(\theta,\sigma)$, $\delta_2(\theta,\sigma)$, $\delta_3(\theta,\sigma)$ from equations (2.176)–(2.178).

2.3.4 The example

Let us consider a system described by equation

$$\begin{cases} \dfrac{dx(t)}{dt} = ax(t) + bx_t(-2h) + cx_t(-h) \\ x(0) = x_0 \\ x_{t=0} = \varphi \end{cases} \tag{2.194}$$

$t \geq 0$, $x(t) \in \mathbb{R}$, $x_t \in L^2([-2h,0),\mathbb{R})$, $x_t(\theta) = x(t+\theta)$, $a,b,c \in \mathbb{R}$, $h > 0$.
The Lyapunov functional is defined by the formula

$$\begin{aligned} V(x(t),x_t) = {} & \alpha x^2(t) + \int_{-2h}^{0} \beta(\theta)x(t)x_t(\theta)d\theta + \int_{-h}^{0} \kappa(\sigma)x(t)x_t(\sigma)d\sigma + \\ & + \int_{-2h}^{0}\int_{\theta}^{0} \delta_1(\theta,\xi)x_t(\theta)x_t(\xi)d\xi d\theta + \int_{-h}^{0}\int_{\sigma}^{0} \delta_2(\sigma,\varsigma)x_t(\sigma)x_t(\varsigma)d\varsigma d\sigma + \\ & + \int_{-2h}^{0}\int_{-h}^{0} \delta_3(\theta,\sigma)x_t(\theta)x_t(\sigma)d\sigma d\theta \end{aligned} \tag{2.195}$$

The set of equations (2.191) becomes

$$\begin{bmatrix} \dfrac{d\beta_1(\xi)}{d\xi} \\ \dfrac{d\beta_2(\xi)}{d\xi} \\ \dfrac{d\eta_1(\xi)}{d\xi} \\ \dfrac{d\eta_2(\xi)}{d\xi} \\ \dfrac{d\kappa(\xi)}{d\xi} \\ \dfrac{d\vartheta(\xi)}{d\xi} \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 & 0 & b \\ c & a & 0 & b & 0 & 0 \\ -b & 0 & -a & -c & 0 & 0 \\ 0 & -b & 0 & -a & -b & 0 \\ 0 & 0 & c & 0 & a & c \\ 0 & -c & 0 & 0 & -c & -a \end{bmatrix} \begin{bmatrix} \beta_1(\xi) \\ \beta_2(\xi) \\ \eta_1(\xi) \\ \eta_2(\xi) \\ \kappa(\xi) \\ \vartheta(\xi) \end{bmatrix} \tag{2.196}$$

for $\xi \in [-h,0]$.
Eigenvalues of the matrix of equation (2.196) are as follows
$\lambda_1 = a$, $\lambda_2 = -a$, $\lambda_3 = \sqrt{g+d}$, $\lambda_4 = -\sqrt{g+d}$, $\lambda_5 = \sqrt{g-d}$, $\lambda_6 = -\sqrt{g-d}$
where $g = a^2 - b^2 - c^2/2$, $d = c\sqrt{c^2/4 + 2b^2 - 2ab}$

Now we give the formulas for determination of the set of initial conditions of equation (2.196). Relations (2.190) take the form as below

$$\begin{cases} \beta_1(-h) = \eta(0) \\ \beta_2(-h) = \beta_1(0) \\ \eta_1(-h) = \beta(0) \\ \eta_2(-h) = \eta_1(0) \\ \kappa(-h) = \vartheta(0) \\ \vartheta(-h) = \kappa(0) \end{cases} \tag{2.197}$$

Among the initial conditions there are relations as below

$$\begin{cases} \beta_1(0) = \eta_1(0) \\ \beta_2(0) = \beta(0) \end{cases} \tag{2.198}$$

Relations (2.193) become

$$\begin{cases} 2a\alpha + \beta(0) + \kappa(0) = -1 \\ 2b\alpha - \eta(0) = 0 \\ 2c\alpha - \vartheta(0) = 0 \\ \eta_1(\xi)\,|_{\xi=-\frac{h}{2}} = \beta_2(\xi)\,|_{\xi=-\frac{h}{2}} \\ \beta_1(\xi)\,|_{\xi=-\frac{h}{2}} = \eta_2(\xi)\,|_{\xi=-\frac{h}{2}} \\ \kappa(\xi)\,|_{\xi=-\frac{h}{2}} = \vartheta(\xi)\,|_{\xi=-\frac{h}{2}} \end{cases} \tag{2.199}$$

Having the solution of equations (2.196)

$\beta_1(\xi)$, $\beta_2(\xi)$, $\eta_1(\xi)$, $\kappa(\xi)$, $\vartheta(\xi)$

for $\xi \in [-h, 0]$ and the matrix α we obtain

$\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$

$\delta_1(\theta,\sigma) = b\eta(\theta - \sigma)$

$\delta_2(\theta,\sigma) = c\vartheta(\theta - \sigma)$

$\delta_3(\theta,\sigma) = c\beta(\theta - \sigma - r_1)$.

Figure 2.5 shows the graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$ and α, obtained with the Matlab code, for given values of parameters a, b and c of the system (2.194).

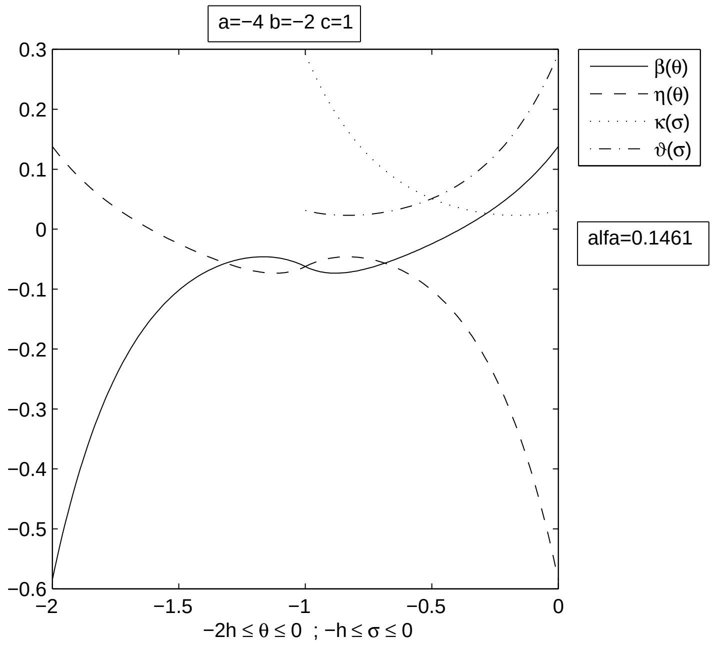

Fig. 2.5. Coefficients of the Lyapunov functional for a system with two delays

2.4 A linear system with both lumped and distributed retarded type time delay

2.4.1 Mathematical model of a linear system with both lumped and distributed retarded type time delay

Let us consider the linear system with both lumped and distributed delay, whose dynamics is described by equation [15]

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx_t(-r) + \displaystyle\int_{-r}^{0} Gx_t(\theta)d\theta \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \varphi \in L^2([-r,0),\mathbb{R}^n) \end{cases} \tag{2.200}$$

for $t \geq t_0$, $r > 0$, where A, B, $G \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $x_t \in L^2([-r,0),\mathbb{R}^n)$, $L^2([-r,0),\mathbb{R}^n)$ is a space of a Lebesgue square integrable functions on interval $[-r,0)$ with values in $\mathbb{R}^n$. The solution of the functional-differential equation (2.200) with initial value (x_0,φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$.

$$x(\cdot,t_0,(x_0,\varphi)) \in W^{1,2}([t_0,\infty),\mathbb{R}^n) \tag{2.201}$$

The function $x_t \in L^2([-r,0),\mathbb{R}^n)$ is a shifted restriction of $x(\cdot,t_0,(x_0,\varphi))$ to the segment $[t-r,t)$.
The state of system (2.200) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \tag{2.202}$$

where $x(t) \in \mathbb{R}^n$,$x_t \in L^2([-r,0),\mathbb{R}^n)$ for $t \geq t_0$
The state space is defined by the formula

$$X = \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n) \tag{2.203}$$

$S = 0$ is the equilibrium point of system (2.200).
In a parametric optimization problem is used an integral quadratic performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t)Wx(t)dt \tag{2.204}$$

where $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix.

2.4.2 Determination of the Lyapunov functional

On the state space X we define a quadratic functional V, positive definite, differentiable, given by the formula [15]

$$V(x(t),x_t) = x^T(t)\alpha x(t) + \int_{-r}^{0} x^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{2.205}$$

for $t \geq t_0$, where $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in C^1([-r,0],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$
$\Omega = \{(\theta,\sigma):\ \theta \in [-r,0],\ \sigma \in [\theta,0]\}$ C^1 is a space of continuous functions with continuous derivative.
In this paragraph we present a procedure of determination of the functional (2.205) coefficients to obtain the Lyapunov functional.
The time derivative of the functional (2.205) on the trajectory of system (2.200) is computed. It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (2.205) and one substitutes in place of $dx(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dx(t)}{dt} = Ax(t) + Bx_t(-r) + \int_{-r}^{0} Gx_t(\theta)d\theta \tag{2.206}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{2.207}$$

In such a manner one attains

$$\frac{dV(x(t),x_t)}{dt} = x^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2}\right]x(t)+$$
$$+x_t^T(-r)\left[2B^T\alpha - \beta^T(-r)\right]x(t)+$$
$$+\int_{-r}^{0} x^T(t)\left[2\alpha G + A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0) + \delta(0,\theta)\right]x_t(\theta)d\theta+$$
$$+\int_{-r}^{0} x_t^T(-r)\left[B^T\beta(\theta) - \delta^T(\theta,-r) - \delta(-r,\theta)\right]x_t(\theta)d\theta+$$
$$-\int_{-r}^{0}\int_{-r}^{0} x_t^T(\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} - G^T\beta(\sigma)\right]x_t(\sigma)d\sigma d\theta \tag{2.208}$$

To achieve negative definiteness of that derivative we assume that the time derivative of the Lyapunov functional V is given as a quadratic form

$$\frac{dV(x(t),x_t)}{dt} \equiv -x^T(t)Wx(t) \tag{2.209}$$

for $t \geq t_0$, where $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix.
When is known the Lyapunov functional and the relationship (2.209) holds, one can easily determine the value of a square indicator of quality of the parametric optimization, because

$$J = \int_{t_0}^{\infty} x^T(t)Wx(t)dt = V(x(t_0),\varphi) \tag{2.210}$$

From equations (2.208) and (2.209) one obtains the set of equations (2.211) to (2.215)

$$A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} = -W \tag{2.211}$$

$$2B^T\alpha - \beta^T(-r) = 0 \tag{2.212}$$

$$A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta(0,\theta) + \delta^T(\theta,0) + 2\alpha G = 0 \tag{2.213}$$

$$B^T\beta(\theta) - \delta(-r,\theta) - \delta^T(\theta,-r) = 0 \tag{2.214}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} - G^T\beta(\sigma) = 0 \tag{2.215}$$

for $\theta \in [-r,0]$, $\sigma \in [-r,0]$.

Let us consider that the solution of equation (2.215) is as below

$$\delta(\theta,\sigma) = f(\theta-\sigma) + f^T(\sigma-\theta) + \int_0^{\sigma} G^T \beta(\xi) d\xi \tag{2.216}$$

where $f \in C^1([-r,r],\mathbb{R}^{n\times n})$.
From equation (2.216) one obtains

$$\delta^T(\theta,0) = f(-\theta) + f^T(\theta) \tag{2.217}$$

$$\delta(0,\theta) = f(-\theta) + f^T(\theta) + \int_0^{\theta} G^T \beta(\xi) d\xi \tag{2.218}$$

$$\delta(-r,\theta) = f(-\theta-r) + f^T(\theta+r) + \int_0^{\theta} G^T \beta(\xi) d\xi \tag{2.219}$$

$$\delta^T(\theta,-r) = f^T(\theta+r) + f(-\theta-r) + \int_0^{-r} \beta^T(\xi) G d\xi \tag{2.220}$$

One puts (2.217) and (2.218) into (2.213), and one gets the formula

$$-\frac{d\beta(\theta)}{d\theta} + A^T\beta(\theta) + 2f^T(\theta) + 2f(-\theta) + \int_0^{\theta} G^T \beta(\xi) d\xi + 2\alpha G = 0 \tag{2.221}$$

for $\theta \in [-r,0]$.
One substitutes (2.219) and (2.220) into (2.214). After some calculations one obtains

$$2f^T(\theta) + 2f(-\theta) = \beta^T(-\theta-r)B - \int_0^{-\theta-r} \beta^T(\xi) G d\xi - \int_0^{-r} G^T \beta(\xi) d\xi \tag{2.222}$$

One puts (2.222) into (2.221). After some calculations one attains

$$\frac{d\beta(\theta)}{d\theta} = A^T\beta(\theta) + \beta^T(-\theta-r)B + \int_{-r}^{\theta} \beta^T(-\xi-r) G d\xi + \int_{-r}^{\theta} G^T \beta(\xi) d\xi + 2\alpha G \tag{2.223}$$

A new function is introduced

$$\kappa(\theta) = \beta(-\theta-r) \tag{2.224}$$

for $\theta \in [-r,0]$.

Now the formula (2.223) can be written in a form

$$\frac{d\beta(\theta)}{d\theta} = A^T\beta(\theta) + \kappa^T(\theta)B + \int_{-r}^{\theta}\kappa^T(\xi)Gd\xi + \int_{-r}^{\theta}G^T\beta(\xi)d\xi + 2\alpha G \tag{2.225}$$

One calculates the derivative of the function κ given by the formula (2.224). The relation (2.225) was taken into account

$$\frac{d\kappa(\theta)}{d\theta} = -A^T\kappa(\theta) - \beta^T(\theta)B + \int_{0}^{\theta}G^T\kappa(\xi)d\xi + \int_{0}^{\theta}\beta^T(\xi)Gd\xi - 2\alpha G \tag{2.226}$$

One introduces two new functions

$$\eta(\theta) = A^T\beta(\theta) + \kappa^T(\theta)B + \int_{-r}^{\theta}\kappa^T(\xi)Gd\xi + \int_{-r}^{\theta}G^T\beta(\xi)d\xi + 2\alpha G \tag{2.227}$$

$$\vartheta(\theta) = -A^T\kappa(\theta) - \beta^T(\theta)B + \int_{0}^{\theta}G^T\kappa(\xi)d\xi + \int_{0}^{\theta}\beta^T(\xi)Gd\xi - 2\alpha G \tag{2.228}$$

Functions η and ϑ are not independent.It is easy to check that they are linked by the formula

$$\eta(-\theta - r) = -\vartheta(\theta) \tag{2.229}$$

for $\theta \in [-r,0]$.
From equations (2.225) and (2.227) it results that

$$\frac{d\beta(\theta)}{d\theta} = \eta(\theta) \tag{2.230}$$

From equations (2.226) and (2.228) it results that

$$\frac{d\kappa(\theta)}{d\theta} = \vartheta(\theta) \tag{2.231}$$

The derivatives of (2.227) and (2.228) are computed. Upon taking the relations (2.230) and (2.231) into account, one gets the formulas

$$\frac{d\eta(\theta)}{d\theta} = A^T\eta(\theta) + \vartheta^T(\theta)B + G^T\beta(\theta) + \kappa^T(\theta)G \tag{2.232}$$

$$\frac{d\vartheta(\theta)}{d\theta} = -A^T\vartheta(\theta) - \eta^T(\theta)B + G^T\kappa(\theta) + \beta^T(\theta)G \tag{2.233}$$

One obtains the system of differential equations

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} = \eta(\theta) \\ \dfrac{d\kappa(\theta)}{d\theta} = \vartheta(\theta) \\ \dfrac{d\eta(\theta)}{d\theta} = A^T\eta(\theta) + \vartheta^T(\theta)B + G^T\beta(\theta) + \kappa^T(\theta)G \\ \dfrac{d\vartheta(\theta)}{d\theta} = -A^T\vartheta(\theta) - \eta^T(\theta)B + G^T\kappa(\theta) + \beta^T(\theta)G \end{cases} \tag{2.234}$$

for $\theta \in [-r, 0]$.
The solution of the differential equations (2.234) satisfies the conditions

$$\beta(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \tag{2.235}$$

$$\eta(\theta)\,|_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\,|_{\theta=-\frac{r}{2}} \tag{2.236}$$

Formula (2.235) was obtained from (2.224) and formula (2.236) from (2.229). Now will be obtained the initial conditions of the differential equations (2.234). From equation (2.224) it results that

$$\kappa(-r) = \beta(0) \tag{2.237}$$

Equation (2.227) implies

$$\eta(-r) = A^T\beta(-r) + \kappa^T(-r)B + 2\alpha G \tag{2.238}$$

Upon taking the relation (2.237) into account, equations (2.211) and (2.212) take the form

$$A^T\alpha + \alpha A + \frac{\kappa(-r) + \kappa^T(-r)}{2} = -W \tag{2.239}$$

$$2B^T\alpha - \beta^T(-r) = 0 \tag{2.240}$$

One obtains the system of algebraic equations

$$\begin{cases} A^T\alpha + \alpha A + \dfrac{\kappa(-r) + \kappa^T(-r)}{2} = -W \\ 2B^T\alpha - \beta^T(-r) = 0 \\ -\eta(-r) + A^T\beta(-r) + \kappa^T(-r)B + 2\alpha G = 0 \\ \beta(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \\ \eta(\theta)\,|_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\,|_{\theta=-\frac{r}{2}} \end{cases} \tag{2.241}$$

The set of algebraic equations (2.241) allows for determination of the matrix α and the initial conditions of system of differential equations (2.234).

From equations (2.221) and (2.230) one obtains

$$f^T(\theta)+f(-\theta)=-\alpha G-\frac{1}{2}A^T\beta(\theta)+\frac{1}{2}\eta(\theta)-\frac{1}{2}\int_0^\theta G^T\beta(\xi)d\xi \tag{2.242}$$

Putting (2.242) into (2.216), one gets the matrix $\delta(\theta,\sigma)$

$$\begin{aligned}\delta(\theta,\sigma)=&-\frac{1}{2}\beta^T(\theta-\sigma)A+\frac{1}{2}\eta^T(\theta-\sigma)+\\&-\frac{1}{2}\int_0^{\theta-\sigma}\beta^T(\xi)Gd\xi+\int_0^\sigma G^T\beta(\xi)d\xi-G^T\alpha\end{aligned} \tag{2.243}$$

for $\theta\in[-r,0]$, $\sigma\in[-r,0]$.
In this way one obtained all coefficients of the functional (2.205). This coefficients depend on the matrices A, B and G of system (2.200). The time derivative of the functional (2.205) is negative definite.

2.4.3 The examples

2.4.3.1 The example 1

Let us consider the system described by equation

$$\begin{cases}\dfrac{dx(t)}{dt}=ax(t)+bx_t(-r)+\int_{-r}^0 gx_t(\theta)d\theta\\ x(t_0)=x_0\in\mathbb{R}\\ x_{t_0}=\varphi\in L^2([-r,0),\mathbb{R})\end{cases} \tag{2.244}$$

$t\geq t_0$, $x(t)\in\mathbb{R}$, $x_t(\theta)=x(t+\theta)$, $\theta\in[-r,0)$, $x_t\in L^2([-r,0),\mathbb{R})$, a, b, $g\in\mathbb{R}$, $r>0$
The Lyapunov functional is defined by the formula

$$V(x(t),x_t)=\alpha x^2(t)+\int_{-r}^0 x(t)\beta(\theta)x_t(\theta)d\theta+\int_{-r}^0\int_{-r}^0 x_t(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{2.245}$$

In a parametric optimization problem is used an integral quadratic performance index of quality

$$J=\int_{t_0}^\infty wx^2(t)dt=V(x_0,\Phi) \tag{2.246}$$

The set of equations (2.234) becomes

$$\begin{bmatrix} \dfrac{d\beta(\theta)}{d\theta} \\ \dfrac{d\eta(\theta)}{d\theta} \\ \dfrac{d\kappa(\theta)}{d\theta} \\ \dfrac{d\vartheta(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ g & a & g & b \\ 0 & 0 & 0 & 1 \\ g & -b & g & -a \end{bmatrix} \begin{bmatrix} \beta(\theta) \\ \eta(\theta) \\ \kappa(\theta) \\ \vartheta(\theta) \end{bmatrix} \tag{2.247}$$

The fundamental matrix of solutions of equation (2.88) is given by

$$R(\theta) = \begin{bmatrix} r_{11}(\theta) & r_{12}(\theta) & r_{13}(\theta) & r_{14}(\theta) \\ r_{21}(\theta) & r_{22}(\theta) & r_{23}(\theta) & r_{24}(\theta) \\ r_{31}(\theta) & r_{32}(\theta) & r_{33}(\theta) & r_{34}(\theta) \\ r_{41}(\theta) & r_{42}(\theta) & r_{43}(\theta) & r_{44}(\theta) \end{bmatrix} \tag{2.248}$$

where

$$r_{11}(\theta) = 1 - \frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\Big[g(bs^2+ag-bg)\theta+ \\ -\frac{g}{2s}(g+bs)(s+a-b)\exp(s\theta) - \frac{g}{2s}(g-bs)(s-a+b)\exp(-s\theta)\Big] \tag{2.249}$$

$$r_{21}(\theta) = \frac{1}{s^2(b^2-ab-g)}\Big[g(bs^2+ag-bg) - \frac{g}{2}(g+bs)(s+a-b)\exp(s\theta)+ \\ +\frac{g}{2}(g-bs)(s-a+b)\exp(-s\theta)\Big] \tag{2.250}$$

$$r_{31}(\theta) = -\frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\Big[-g(bs^2+ag-bg)\theta+ \\ -\frac{g}{2s}(s^2-as-g)(s+a-b)\exp(s\theta) - \frac{g}{2s}(s^2+as-g)(s-a+b)\exp(-s\theta)\Big] \tag{2.251}$$

$$r_{41}(\theta) = \frac{1}{s^2(b^2-ab-g)}\Big[-g(bs^2+ag-bg)+ \\ -\frac{g}{2}(s^2-as-g)(s+a-b)\exp(s\theta) + \frac{g}{2}(s^2+as-g)(s-a+b)\exp(-s\theta)\Big] \tag{2.252}$$

$$r_{12}(\theta) = -\frac{a}{s^2} + \frac{g}{s^2}\theta + \frac{1}{s^2(b^2-ab-g)}\left[-\frac{1}{2s}(g+bs)(a^2+as-bs-ab+g)\exp(s\theta)+ \right.$$
$$\left. +\frac{1}{2s}(g-bs)(a^2-as+bs-ab+g)\exp(-s\theta)\right] \tag{2.253}$$

$$r_{22}(\theta) = \frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\left[-\frac{1}{2}(g+bs)(a^2+as-bs-ab+g)\exp(s\theta)+ \right.$$
$$\left. -\frac{1}{2}(g-bs)(a^2-as+bs-ab+g)\exp(-s\theta)\right] \tag{2.254}$$

$$r_{32}(\theta) = \frac{b}{s^2} - \frac{g}{s^2}\theta + \frac{1}{s^2(b^2-ab-g)}\left[-\frac{1}{2s}(s^2-as-g)(a^2+as-bs-ab+g)\exp(s\theta)+ \right.$$
$$\left. +\frac{1}{2s}(s^2+as-g)(a^2-as+bs-ab+g)\exp(-s\theta)\right] \tag{2.255}$$

$$r_{42}(\theta) = -\frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\left[-\frac{1}{2}(s^2-as-g)(a^2+as-bs-ab+g)\exp(s\theta)+ \right.$$
$$\left. -\frac{1}{2}(s^2+as-g)(a^2-as+bs-ab+g)\exp(-s\theta)\right] \tag{2.256}$$

$$r_{13}(\theta) = -\frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\left[g(bs^2+ag-bg)\theta - \frac{g}{2s}(g+bs)(s+a-b)\exp(s\theta)+ \right.$$
$$\left. -\frac{g}{2s}(g-bs)(s-a+b)\exp(-s\theta)\right] \tag{2.257}$$

$$r_{23}(\theta) = \frac{1}{s^2(b^2-ab-g)}\left[g(bs^2+ag-bg) - \frac{g}{2}(g+bs)(s+a-b)\exp(s\theta)+ \right.$$
$$\left. +\frac{g}{2}(g-bs)(s-a+b)\exp(-s\theta)\right] \tag{2.258}$$

$$r_{33}(\theta) = 1 - \frac{g}{s^2} + \frac{1}{s^2(b^2-ab-g)}\left[-g(bs^2+ag-bg)\theta+ \right.$$
$$\left. -\frac{g}{2s}(s^2-as-g)(s+a-b)\exp(s\theta) - \frac{g}{2s}(s^2+as-g)(s-a+b)\exp(-s\theta)\right] \tag{2.259}$$

$$r_{43}(\theta) = \frac{1}{s^2(b^2-ab-g)}\left[-g(bs^2+ag-bg) - \frac{g}{2}(s^2-as-g)(s+a-b)\exp(s\theta)+ \right.$$
$$\left. +\frac{g}{2}(s^2+as-g)(s-a+b)\exp(-s\theta)\right] \tag{2.260}$$

$$r_{14}(\theta) = -\frac{b}{s^2} - \frac{g}{s^2}\theta + \frac{1}{2s^3}(g+bs)\exp(s\theta) - \frac{1}{2s^3}(g-bs)\exp(-s\theta) \tag{2.261}$$

$$r_{24}(\theta) = -\frac{g}{s^2} + \frac{1}{2s^2}(g+bs)\exp(s\theta) + \frac{1}{2s^2}(g-bs)\exp(-s\theta) \tag{2.262}$$

$$r_{34}(\theta) = \frac{a}{s^2} + \frac{g}{s^2}\theta + \frac{1}{2s^3}(s^2-as-g)\exp(s\theta) - \frac{1}{2s^3}(s^2+as-g)\exp(-s\theta) \tag{2.263}$$

$$r_{44}(\theta) = \frac{g}{s^2} + \frac{1}{2s^2}(s^2-as-g)\exp(s\theta) + \frac{1}{2s^2}(s^2+as-g)\exp(-s\theta) \tag{2.264}$$

where

$$s = \sqrt{a^2 - b^2 + 2g} \tag{2.265}$$

The solution of the set of equations (2.247) is given in a form

$$\beta(\theta) = r_{11}(\theta+r)\beta(-r) + r_{12}(\theta+r)\eta(-r) + r_{13}(\theta+r)\kappa(-r) + r_{14}(\theta+r)\vartheta(-r) \tag{2.266}$$

$$\eta(\theta) = r_{21}(\theta+r)\beta(-r) + r_{22}(\theta+r)\eta(-r) + r_{23}(\theta+r)\kappa(-r) + r_{24}(\theta+r)\vartheta(-r) \tag{2.267}$$

$$\kappa(\theta) = r_{31}(\theta+r)\beta(-r) + r_{32}(\theta+r)\eta(-r) + r_{33}(\theta+r)\kappa(-r) + r_{34}(\theta+r)\vartheta(-r) \tag{2.268}$$

$$\vartheta(\theta) = r_{41}(\theta+r)\beta(-r) + r_{42}(\theta+r)\eta(-r) + r_{43}(\theta+r)\kappa(-r) + r_{44}(\theta+r)\vartheta(-r) \tag{2.269}$$

Now will be given the formulas for determination of the set of the initial conditions of equation (2.247) and the coefficient α

$$\begin{cases} 2a\alpha + \kappa(-r) = -w \\ 2b\alpha - \beta(-r) = 0 \\ -\eta(-r) + a\beta(-r) + b\kappa(-r) + 2g\alpha = 0 \\ \beta(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \\ \eta(\theta)\,|_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\,|_{\theta=-\frac{r}{2}} \end{cases} \tag{2.270}$$

The set of algebraic equations (2.270) can be written in the equivalent form

$$\kappa(-r) = -w - 2a\alpha \tag{2.271}$$

$$\beta(-r) = 2b\alpha \tag{2.272}$$

$$\eta(-r) = (2g - bw)\alpha \tag{2.273}$$

$$2p_{11}\alpha + p_{12}\vartheta(-r) = p_{13}w \tag{2.274}$$

$$2p_{21}\alpha + p_{22}\vartheta(-r) = p_{23}w \tag{2.275}$$

where

$$p_{11} = (s^2 - g)(a + b - gr) - \frac{g}{2s}\left(a^2 - b^2 - as - bs\right)\exp\left(-\frac{sr}{2}\right) + \\ + \frac{g}{2s}\left(a^2 - b^2 + as + bs\right)\exp\left(\frac{sr}{2}\right) \tag{2.276}$$

$$p_{12} = -a - b + gr - \frac{1}{2s}\left(a^2 - b^2 - as - bs\right)\exp\left(-\frac{sr}{2}\right) + \\ + \frac{1}{2s}\left(a^2 - b^2 + as + bs\right)\exp\left(\frac{sr}{2}\right) \tag{2.277}$$

$$p_{13} = -s^2 - ab - b^2 + agr + \frac{1}{2s}\left(bs^2 + b^2 s + abs + 2ag\right)\exp\left(-\frac{sr}{2}\right) + \\ - \frac{1}{2s}\left(bs^2 - b^2 s - abs + 2ag\right)\exp\left(\frac{sr}{2}\right) \tag{2.278}$$

$$p_{21} = \frac{gs}{2}\left(s - a + b\right)\exp\left(-\frac{sr}{2}\right) + \frac{gs}{2}\left(s + a - b\right)\exp\left(\frac{sr}{2}\right) \tag{2.279}$$

$$p_{22} = \frac{s}{2}\left(s - a + b\right)\exp\left(-\frac{sr}{2}\right) + \frac{s}{2}\left(s + a - b\right)\exp\left(\frac{sr}{2}\right) \tag{2.280}$$

$$p_{23} = \frac{s}{2}\left(s^2 - a^2 + ab - bs\right)\exp\left(-\frac{sr}{2}\right) - \frac{s}{2}\left(s^2 - a^2 + ab - bs\right)\exp\left(\frac{sr}{2}\right) \tag{2.281}$$

The parameter α is given by a term

$$\alpha = \frac{1}{M}\Big[-aw(a+b)(b^2 - ab - g) + \\ - \frac{w}{2}\left(s^2 - a^2 + ab - bs\right)(a^2 - b^2) + \frac{bsw}{2}\left(a + b\right)(s - a + b)\exp(-sr) + \\ + \frac{sw}{2}\left(-s^3 - a^3 - b^3 - 2b^2 s - 2abs + a^2 b + ab^2 - grs(s - a - b)\right)\exp\left(-\frac{sr}{2}\right) + \\ + \frac{sw}{2}\left(-s^3 + a^3 + b^3 - 2as^2 - a^2 b - ab^2 + grs(s + a - b)\right)\exp\left(\frac{sr}{2}\right)\Big] \tag{2.282}$$

where

$$M = s^3(a+b-gr)(s-a+b)\exp\Big(-\frac{sr}{2}\Big)+$$
$$+s^3(a+b-gr)(s+a-b)\exp\Big(\frac{sr}{2}\Big) \tag{2.283}$$

Having the solution of equations (2.247) and the coefficient α one obtains $\delta(\theta,\sigma)$

$$\delta(\theta,\sigma) = -ga - \frac{1}{2}a\beta(\theta-\sigma) + \frac{1}{2}\eta(\theta-\sigma)+$$
$$-\frac{1}{2}\int_0^{\theta-\sigma} g\beta(\xi)d\xi + \int_0^{\sigma} g\beta(\xi)d\xi \tag{2.284}$$

Figure 2.6 shows graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\theta)$, $\vartheta(\theta)$ and α, obtained with the Matlab code, for given values of parameters a, b, g, w, r of system (2.177).

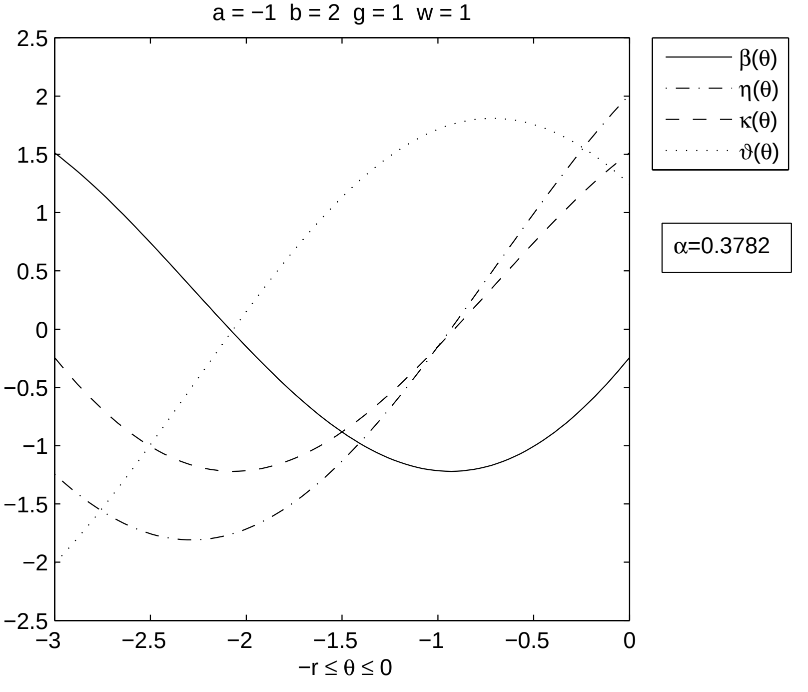

Fig. 2.6. Coefficients of the Lyapunov functional with distributed delay

2.4.3.2 The example 2

Let us consider the system described by the equation

$$\begin{cases} \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1(t-r) \\ x_2(t-r) \end{bmatrix} + \\ + \int_{-r}^{0} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{bmatrix} d\theta \\ \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ \begin{bmatrix} x_1(t_0+\theta) \\ x_2(t_0+\theta) \end{bmatrix} = \begin{bmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{bmatrix} \end{cases} \tag{2.285}$$

The Lyapunov functional is defined by the formula

$$V(x_1(t),x_2(t),x_1(t+\cdot),x_2(t+\cdot)) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} +$$

$$+ \int_{-r}^{0} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \beta_{11}(\theta) & \beta_{12}(\theta) \\ \beta_{21}(\theta) & \beta_{22}(\theta) \end{bmatrix} \begin{bmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{bmatrix} d\theta +$$

$$+ \int_{-r}^{0} \int_{-r}^{0} \begin{bmatrix} x_1(t+\theta) & x_2(t+\theta) \end{bmatrix} \begin{bmatrix} \delta_{11}(\theta,\sigma) & \delta_{12}(\theta,\sigma) \\ \delta_{21}(\theta,\sigma) & \delta_{22}(\theta,\sigma) \end{bmatrix} \begin{bmatrix} x_1(t+\sigma) \\ x_2(t+\sigma) \end{bmatrix} d\sigma d\theta \tag{2.286}$$

The set of equations (2.167) becomes

$$\frac{d}{d\theta} \begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} = Q \begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} \tag{2.287}$$

for $\theta \in [-r,0]$, where

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \tag{2.288}$$

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g_{11} & g_{21} & 0 & 0 & a_{11} & a_{21} & 0 & 0 \\ g_{12} & g_{22} & 0 & 0 & a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & a_{11} & a_{21} \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & a_{12} & a_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{11} & g_{21} & 0 & 0 & -b_{11} & -b_{21} & 0 & 0 \\ 0 & g_{21} & g_{11} & 0 & 0 & 0 & -b_{11} & -b_{21} \\ g_{12} & g_{22} & 0 & 0 & -b_{12} & -b_{22} & 0 & 0 \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & -b_{12} & -b_{22} \end{bmatrix} \tag{2.289}$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{11} & g_{21} & 0 & 0 & b_{11} & b_{21} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & b_{11} & b_{21} \\ g_{12} & g_{22} & 0 & 0 & b_{12} & b_{22} & 0 & 0 \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & b_{12} & b_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g_{11} & g_{21} & 0 & 0 & -a_{11} & -a_{21} & 0 & 0 \\ g_{12} & g_{22} & 0 & 0 & -a_{12} & -a_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & -a_{11} & -a_{21} \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & -a_{12} & -a_{22} \end{bmatrix} \tag{2.290}$$

$$\begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} = e^{Q(\theta+r)} \begin{bmatrix} col\,\beta(-r) \\ col\,\eta(-r) \\ col\,\kappa(-r) \\ col\,\vartheta(-r) \end{bmatrix} \tag{2.291}$$

for $\theta \in [-r, 0]$.

We introduce

$$e^{Q\frac{r}{2}} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \tag{2.292}$$

Now we give the formulas for determination of the set of the initial conditions of equation (2.287) and the matrix α.

$$Z \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{22} \\ col\,\beta(-r) \\ col\,\eta(-r) \\ col\,\kappa(-r) \\ col\,\vartheta(-r) \end{bmatrix} = \begin{bmatrix} -w_{11} \\ -w_{12} \\ -w_{22} \\ 0_{(16,1)} \end{bmatrix} \tag{2.293}$$

where

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ 0_{(8,3)} & Z_{22} & Z_{23} \end{bmatrix} \tag{2.294}$$

$$Z_{11} = \begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11}+a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \\ b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & b_{11} & b_{21} \\ 0 & b_{12} & b_{22} \\ 2g_{11} & 2g_{21} & 0 \\ 2g_{12} & 2g_{22} & 0 \\ 0 & 2g_{11} & 2g_{21} \\ 0 & 2g_{12} & 2g_{22} \end{bmatrix} \tag{2.295}$$

$$Z_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ a_{11} & a_{21} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{21} & 0 & 0 & -1 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & a_{12} & a_{22} & 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.296}$$

$$Z_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{11} & b_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & b_{22} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.297}$$

$$Z_{22} = \begin{bmatrix} p_{11} - p_{31} & p_{12} - p_{32} \\ p_{21} + p_{41} & p_{22} + p_{42} \end{bmatrix} \tag{2.298}$$

$$Z_{23} = \begin{bmatrix} p_{13} - p_{33} & p_{14} - p_{34} \\ p_{23} + p_{43} & p_{24} + p_{44} \end{bmatrix} \tag{2.299}$$

Now we obtain the matrix $\delta(\theta,\sigma)$

$$\delta_{11}(\theta,\sigma) = -\frac{1}{2}a_{11}\beta_{11}(\theta-\sigma) - \frac{1}{2}a_{21}\beta_{21}(\theta-\sigma) + \frac{1}{2}\eta_{11}(\theta-\sigma) +$$
$$-\frac{1}{2}\int_0^{\theta-\sigma}\Big[g_{11}\beta_{11}(\xi) + g_{21}\beta_{21}(\xi)\Big]d\xi + \int_0^{\sigma}\Big[g_{11}\beta_{11}(\xi) + g_{21}\beta_{21}(\xi)\Big]d\xi +$$
$$-g_{11}\alpha_{11} - g_{21}\alpha_{12} \tag{2.300}$$

$$\delta_{12}(\theta,\sigma) = -\frac{1}{2}a_{12}\beta_{11}(\theta-\sigma) - \frac{1}{2}a_{22}\beta_{21}(\theta-\sigma) + \frac{1}{2}\eta_{21}(\theta-\sigma) +$$
$$-\frac{1}{2}\int_0^{\theta-\sigma}\Big[g_{12}\beta_{11}(\xi) + g_{22}\beta_{21}(\xi)\Big]d\xi + \int_0^{\sigma}\Big[g_{11}\beta_{12}(\xi) + g_{21}\beta_{22}(\xi)\Big]d\xi +$$
$$-g_{11}\alpha_{12} - g_{21}\alpha_{22} \tag{2.301}$$

$$\delta_{21}(\theta,\sigma) = -\frac{1}{2}a_{11}\beta_{12}(\theta-\sigma) - \frac{1}{2}a_{21}\beta_{22}(\theta-\sigma) + \frac{1}{2}\eta_{12}(\theta-\sigma) +$$
$$-\frac{1}{2}\int_0^{\theta-\sigma}\Big[g_{11}\beta_{12}(\xi) + g_{21}\beta_{22}(\xi)\Big]d\xi + \int_0^{\sigma}\Big[g_{12}\beta_{11}(\xi) + g_{22}\beta_{21}(\xi)\Big]d\xi +$$
$$-g_{12}\alpha_{11} - g_{22}\alpha_{12} \tag{2.302}$$

$$\delta_{22}(\theta,\sigma) = -\frac{1}{2}a_{12}\beta_{12}(\theta-\sigma) - \frac{1}{2}a_{22}\beta_{22}(\theta-\sigma) + \frac{1}{2}\eta_{22}(\theta-\sigma) +$$
$$-\frac{1}{2}\int_0^{\theta-\sigma}\Big[g_{12}\beta_{12}(\xi) + g_{22}\beta_{22}(\xi)\Big]d\xi + \int_0^{\sigma}\Big[g_{12}\beta_{12}(\xi) + g_{22}\beta_{22}(\xi)\Big]d\xi +$$
$$-g_{12}\alpha_{12} - g_{22}\alpha_{22} \tag{2.303}$$

Figures 2.7–2.10 show graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\theta)$, $\vartheta(\theta)$ obtained with the Matlab code, for given values of matrices A, B, G, W of system (2.285)

$$A = \begin{bmatrix} -1 & 0.3 \\ 0.5 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0.4 \\ 0.1 & 2 \end{bmatrix}$$
$$G = \begin{bmatrix} 1 & 0.7 \\ 0.3 & 2 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.304}$$

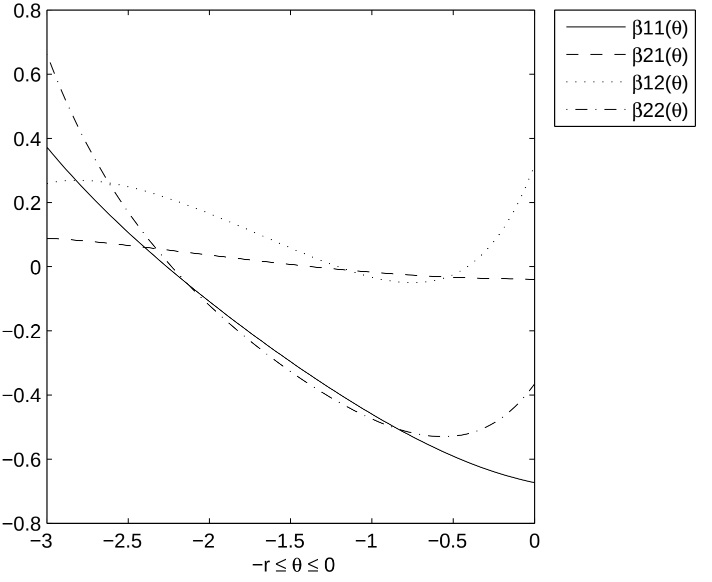

Fig. 2.7. Elements of matrix $\beta(\theta)$

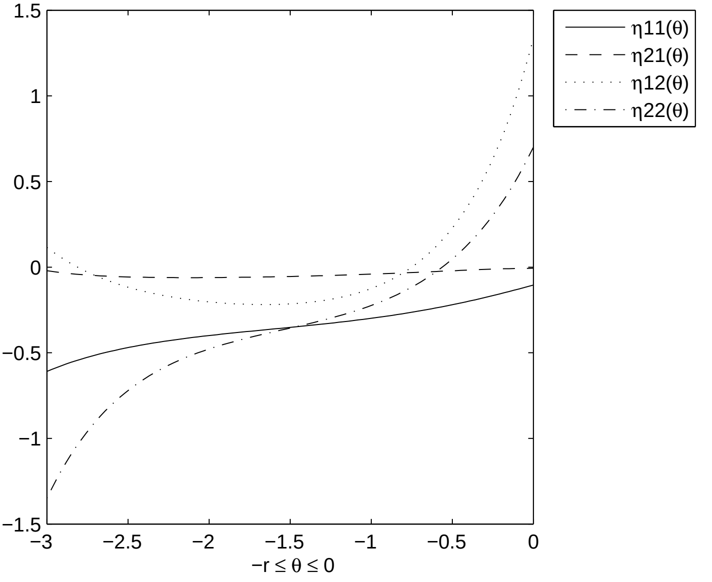

Fig. 2.8. Elements of matrix $\eta(\theta)$

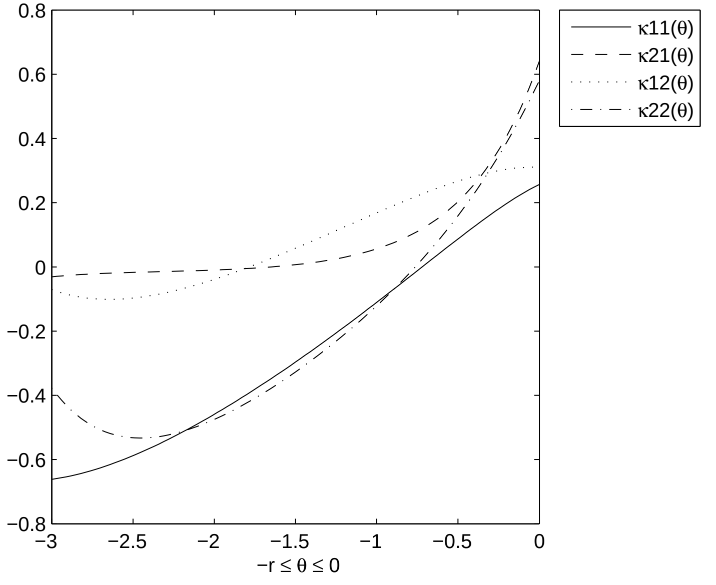

Fig. 2.9. Elements of matrix $\kappa(\theta)$

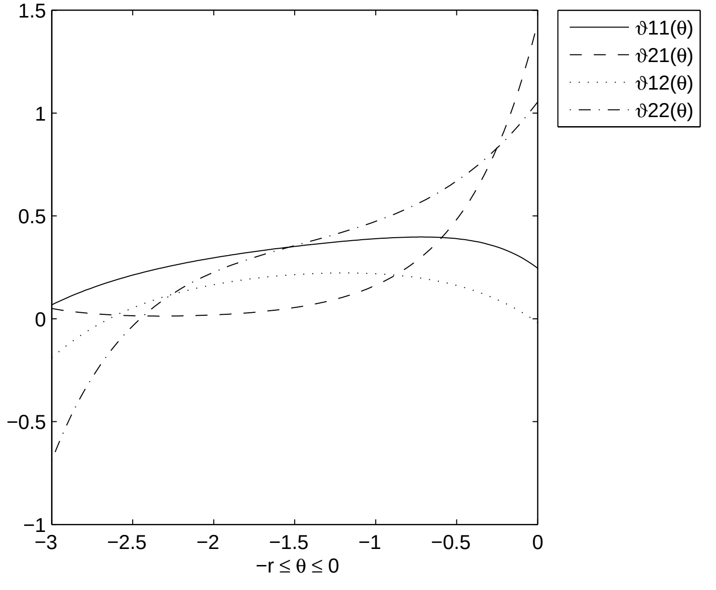

Fig. 2.10. Elements of matrix $\vartheta(\theta)$

Matrix α obtained for the values (2.304) is given below

$$\alpha = \begin{bmatrix} 0.1833 & 0.0281 \\ 0.0281 & 0.1600 \end{bmatrix} \tag{2.305}$$

2.5 A linear system with a retarded type time-varying delay

2.5.1 Mathematical model of a linear system with a retarded type time-varying delay

Let us consider a linear system with a retarded type time-varying delay, whose dynamics is described by the equation [16]

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx(t - \tau(t)) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x(t_0 + \theta) = \varphi(\theta) \end{cases} \tag{2.306}$$

for $t \geq t_0$, $\theta \in [-r,0)$,where A, $B \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$,$\varphi \in L^2([-r,0),\mathbb{R}^n)$, $\tau(t)$ is a time-varying delay satisfying the condition $0 \leq \tau(t) \leq r$; $d\tau(t)/dt \neq 1$; $t \geq t_0$ where r is positive constant. $L^2([-r,0),\mathbb{R}^n)$ is a space of Lebesgue square integrable functions on interval $[-r,0)$ with values in $\mathbb{R}^n$.
Using the formula (2.5) one can write the equation (2.306) in a form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx_t(-\tau(t)) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \varphi \in L^2([-r,0),\mathbb{R}^n) \end{cases} \tag{2.307}$$

The solution of the functional-differential equation (2.307) with initial value (x_0, φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$.

$$x(\cdot, t_0, (x_0, \varphi)) \in W^{1,2}([t_0, \infty), \mathbb{R}^n) \tag{2.308}$$

The function $x_t \in L^2([-r,0),\mathbb{R}^n)$ is a shifted restriction of $x(\cdot, t_0, (x_0, \varphi))$ to the segment $[t - r, t)$.
The state of system (2.307) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \tag{2.309}$$

for $t \geq t_0$.

The state space is defined by the formula

$$X = \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n) \tag{2.310}$$

In a parametric optimization problem is used an integral quadratic performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t)Wx(t)dt \tag{2.311}$$

where $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix.

2.5.2 Determination of the Lyapunov functional

On the state space X we define a quadratic functional V, positive definite, differentiable, given by the formula [16]

$$V(x(t),x_t,t) = x^T(t)\alpha(t)x(t) + \int_{-\tau(t)}^{0} x^T(t)\beta(\theta+\tau(t))x_t(\theta)d\theta +$$
$$+ \int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta+\tau(t),\sigma+\tau(t))x_t(\sigma)d\sigma d\theta \tag{2.312}$$

for $t \geq t_0$, where $\alpha \in C^1([t_0,\infty),\mathbb{R}^{n\times n})$, $\alpha(t)$ is positively defined,
$\beta \in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$, $\Omega = \{(\theta,\sigma):\ \theta \in [0,\tau(t)],\ \sigma \in [\theta,0]\}$,
$0 \leq \tau(t) \leq r$.
C^1 is a space of continuous functions with continuous derivative.
In this paragraph is given a procedure of determination of the functional (2.312) coefficients to obtain the Lyapunov functional.
The time derivative of the functional (2.312) on the trajectory of system (2.307) is computed. It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (2.312) and one substitutes in place of $dx(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dx(t)}{dt} = Ax(t) + Bx_t(-\tau(t)) \tag{2.313}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{2.314}$$

In such a manner one attains

$$\frac{dV(x(t),x_t,t)}{dt} = x^T(t)\Big[A^T\alpha(t) + \alpha(t)A + \frac{d\alpha(t)}{dt} + \beta(\tau(t))\Big]x(t) +$$
$$+ x_t^T(-\tau(t))\Big[B^T\left(\alpha(t)+\alpha^T(t)\right) + \beta^T(0)\Big(\frac{d\tau(t)}{dt} - 1\Big)\Big]x(t) +$$

$$+\int_{-\tau(t)}^{0} x^T(t)\Big[A^T\beta(\theta+\tau(t))+\frac{d\beta(\theta+\tau(t))}{dt}-\frac{d\beta(\theta+\tau(t))}{d\theta}+$$

$$+\delta^T(\theta+\tau(t),\tau(t))\Big]x_t(\theta)d\theta+\int_{-\tau(t)}^{0} x_t^T(-\tau(t))\Big[B^T\beta(\theta+\tau(t))+$$

$$+\delta(0,\theta+\tau(t))\left(\frac{d\tau(t)}{dt}-1\right)\Big]x_t(\theta)d\theta+\int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta)\Big[\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt}+$$

$$-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma}\Big]x_t(\sigma)d\sigma d\theta \tag{2.315}$$

for $t\geq t_0$ where $\alpha\in C^1([t_0,\infty),\mathbb{R}^{n\times n})$, $\beta\in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$, $\delta\in C^1(\Omega,\mathbb{R}^{n\times n})$,
$\Omega=\{(\theta,\sigma):\ \theta\in[0,\tau(t)],\sigma\in[\theta,0]\}$, $0\leq\tau(t)\leq r$.

To achieve negative definiteness of that derivative we assume that the time derivative (2.315) satisfies the relationship

$$\frac{dV(x(t),x_t,t)}{dt}\equiv -x^T(t)Wx(t) \tag{2.316}$$

for $t\geq t_0$, where $W\in\mathbb{R}^{n\times n}$ is positive definite matrix.
From equations (2.315) and (2.316) the set of equations is obtained

$$A^T\alpha(t)+\alpha(t)A+\frac{d\alpha(t)}{dt}+\beta(\tau(t))=-W \tag{2.317}$$

$$B^T\left(\alpha(t)+\alpha^T(t)\right)+\beta^T(0)\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{2.318}$$

$$A^T\beta(\theta+\tau(t))+\frac{d\beta(\theta+\tau(t))}{dt}-\frac{d\beta(\theta+\tau(t))}{d\theta}+\delta^T(\theta+\tau(t),\tau(t))=0 \tag{2.319}$$

$$B^T\beta(\theta+\tau(t))+\delta(0,\theta+\tau(t))\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{2.320}$$

$$\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma}=0 \tag{2.321}$$

for $t\geq t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$ where $0\leq\tau(t)\leq r$.

The new variables are introduced

$$\xi=\theta+\tau(t) \tag{2.322}$$

$$\eta=\sigma+\tau(t) \tag{2.323}$$

One calculates the derivatives

$$\frac{d\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{dt}=\frac{d\delta(\xi,\eta)}{dt}=\frac{\partial\delta(\xi,\eta)}{\partial\xi}\frac{d\tau(t)}{dt}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}\frac{d\tau(t)}{dt} \tag{2.324}$$

$$\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\theta}=\frac{\partial\delta(\xi,\eta)}{\partial\theta}=\frac{\partial\delta(\xi,\eta)}{\partial\xi} \tag{2.325}$$

$$\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\sigma}=\frac{\partial\delta(\xi,\eta)}{\partial\sigma}=\frac{\partial\delta(\xi,\eta)}{\partial\eta} \tag{2.326}$$

$$\frac{d\beta\left(\theta+\tau(t)\right)}{dt}=\frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial t}=\frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt} \tag{2.327}$$

$$\frac{d\beta\left(\theta+\tau(t)\right)}{d\theta}=\frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial\theta}=\frac{d\beta(\xi)}{d\xi} \tag{2.328}$$

The formula (2.321) takes a form

$$\frac{\partial\delta(\xi,\eta)}{\partial\xi}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}=0 \tag{2.329}$$

for $t\geq t_0$, $\theta\in[-\tau(t),0]$, $\sigma\in[\theta,0]$, $\xi\in[0,\tau(t)]$, $\eta\in[\xi,\tau(t)]$ where $0\leq\tau(t)\leq r$.
The solution of equation (2.321) is given by the formula

$$\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)=\delta\left(\xi,\eta\right)=f(\xi-\eta)=f\left(\theta-\sigma\right) \tag{2.330}$$

for $t\geq t_0$, $\theta\in[-\tau(t),0]$, $\sigma\in[\theta,0]$, $0\leq\tau(t)\leq r$ where $f\in C^1\left([-r,r],\mathbb{R}^{n\times n}\right)$
Taking into account the formula (2.330) one gets from equation (2.320) the relationship

$$\delta\left(0,\theta+\tau(t)\right)=f\left(-\tau(t)-\theta\right)=\left(1-\frac{d\tau(t)}{dt}\right)^{-1}B^T\beta\left(\theta+\tau(t)\right) \tag{2.331}$$

Hence

$$f\left(\xi\right)=\left(1-\frac{d\tau(t)}{dt}\right)^{-1}B^T\beta\left(-\xi\right) \tag{2.332}$$

for $\xi\in[0,\tau(t)]$ where $0\leq\tau(t)\leq r$
Formula (2.330) implies

$$\delta^T\left(\theta+\tau(t),\tau(t)\right)=f^T\left(\theta\right)=\left(1-\frac{d\tau(t)}{dt}\right)^{-1}\beta^T\left(-\theta\right)B \tag{2.333}$$

After putting the term (2.333) into the formula (2.319) one obtains a relationship

$$A^T\beta\left(\theta+\tau(t)\right)+\frac{d\beta\left(\theta+\tau(t)\right)}{dt}-\frac{d\beta\left(\theta+\tau(t)\right)}{d\theta}+\left(1-\frac{d\tau(t)}{dt}\right)^{-1}\beta^T\left(-\theta\right)B=0 \tag{2.334}$$

Taking into account the formulas (2.322), (2.327) and (2.328) one obtains from equation (2.334) the relationship

$$\frac{d\beta(\xi)}{d\xi} = -\left(\frac{d\tau(t)}{dt} - 1\right)^{-1} A^T \beta(\xi) + \left(\frac{d\tau(t)}{dt} - 1\right)^{-2} \beta^T(-\xi + \tau(t)) B \tag{2.335}$$

for $\xi \in [0, \tau(t)]$ where $0 \le \tau(t) \le r$
Using the relationship (2.335) the derivative of the term $\beta(-\xi + \tau(t))$ with respect to ξ is calculated. One obtains

$$\frac{d\beta(-\xi + \tau(t))}{d\xi} = -\left(\frac{d\tau(t)}{dt} - 1\right)^{-2} \beta^T(\xi) B + \left(\frac{d\tau(t)}{dt} - 1\right)^{-1} A^T \beta(-\xi + \tau(t)) \tag{2.336}$$

for $\xi \in [0, \tau(t)]$ where $0 \le \tau(t) \le r$
In such a way one attains the set of differential equations

$$\begin{cases} \dfrac{d\beta(\xi)}{d\xi} = -\left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} A^T \beta(\xi) + \\ \qquad + \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-2} \beta^T(-\xi + \tau(t)) B \\ \dfrac{d\beta(-\xi + \tau(t))}{d\xi} = -\left(\dfrac{d\tau(t)}{dt} - 1\right)^{-2} \beta^T(\xi) B + \\ \qquad + \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} A^T \beta(-\xi + \tau(t)) \end{cases} \tag{2.337}$$

for each fixed $t \ge t_0$, $\xi \in [0, \tau(t)]$ where $0 \le \tau(t) \le r$ with the initial conditions $\beta(0)$ and $\beta(\tau(t))$.
There holds the relationship between $\beta(\xi)$ and $\beta(-\xi + \tau(t))$

$$\beta(\xi)\,|_{\xi = \frac{\tau(t)}{2}} = \beta(-\xi + \tau(t))\,|_{\xi = \frac{\tau(t)}{2}} \tag{2.338}$$

The derivative of equation (2.318) with respect to t is calculated

$$B^T \left(\frac{d\alpha(t)}{dt} + \frac{d\alpha^T(t)}{dt}\right) + \frac{d\beta^T(0)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right) + \beta^T(0)\frac{d^2\tau(t)}{dt^2} = 0 \tag{2.339}$$

From equation (2.335) it results that

$$\frac{d\beta^T(0)}{dt} = -\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)^{-1} \beta^T(0) A + \frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)^{-2} B^T \beta(\tau(t)) \tag{2.340}$$

The equation (2.317) implies

$$\frac{d\alpha(t)}{dt} = -A^T \alpha(t) - \alpha(t) A - \beta(\tau(t)) - W \tag{2.341}$$

One puts the terms (2.340) and (2.341) into equation (2.339). After calculations one attains

$$B^T\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right]+\beta^T(0)\left(\frac{d\tau(t)}{dt}A-\frac{d^2\tau(t)}{dt^2}I\right)+$$

$$-\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))+B^T\beta^T(\tau(t))=-B^T\left(W+W^T\right) \tag{2.342}$$

Solving the set of equations (2.342), (2.318) and (2.338) one obtains the matrix $\alpha(t)$ and the initial conditions of system (2.337). That set of equations is written below

$$B^T\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right]+\beta^T(0)\left(\frac{d\tau(t)}{dt}A-\frac{d^2\tau(t)}{dt^2}I\right)+$$

$$-\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))+B^T\beta^T(\tau(t))=-B^T\left(W+W^T\right) \tag{2.343}$$

$$B^T\left(\alpha(t)+\alpha^T(t)\right)+\beta^T(0)\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{2.344}$$

$$\beta(\xi)\,|_{\xi=\frac{\tau(t)}{2}}=\beta(-\xi+\tau(t))\,|_{\xi=\frac{\tau(t)}{2}} \tag{2.345}$$

Having the solution of the set of differential equations (2.337) and taking into account the formulas (2.322), (2.330) and (2.332) one can get the matrices

$$\beta(\theta+\tau(t))=\beta(\xi)\,|_{\xi=\theta+\tau(t)} \tag{2.346}$$

$$\delta(\theta+\tau(t),\sigma+\tau(t))=\left(1-\frac{d\tau(t)}{dt}\right)^{-1}B^T\beta(\sigma-\theta) \tag{2.347}$$

for $t\geq t_0$, $\theta\in[-\tau(t),0]$, $\sigma\in[\theta,0]$ where $0\leq\tau(t)\leq r$.
In this way one obtained all coefficients of the functional (2.312). This coefficients depend on the matrices A and B of system (2.307). The time derivative of the functional (2.312) is negative definite.

2.5.3 The examples

2.5.3.1 Inertial system with delay and a P controller

Let us consider a first order inertial system with delay described by the equation

$$\begin{cases}\frac{dx(t)}{dt}=-\frac{q}{T}x(t)+\frac{k_0}{T}u(t-\tau(t))\\ x(t_0)=x_o\\ x(t_0+\theta)=\varphi(\theta)\\ u(t)=-px(t)\end{cases} \tag{2.348}$$

$t\geq t_0$, $x(t)\in\mathbb{R}$, $\varphi\in W^{1,2}([-r,0),\mathbb{R})$, $\theta\in[-r,0)$, p, k_0, T, q, $x_0\in\mathbb{R}$, $r\geq 0$, $\tau(t)$ is a time-varying delay satisfying the condition $0\leq\tau(t)\leq r$; $d\tau(t)/dt\neq 1$; $t\geq t_0$ where r is positive

constant. The parameter k_0 is a gain of a plant, p is a gain of a P controller, T is a system time constant, x_0 is an initial state of system. In the case $q=1$ an equation (2.348) describes a static object and in the case $q=0$ an equation (2.348) describes an astatic object.
One can reshape an equation (2.348) to a form

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{q}{T}x(t) - \frac{k_0 p}{T}x(t-\tau(t)) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{2.349}$$

Using the formula (3.7) one can write the equation (2.349) in a form

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{q}{T}x(t) - \frac{k_0 p}{T}x_t(-\tau(t)) \\ x(t_0) = x_0 \\ x_{t_0} = \varphi \end{cases} \tag{2.350}$$

The Lyapunov functional is given by the formula

$$V(x(t), x_t, t) = \alpha(t)x^2(t) + \int_{-\tau(t)}^{0} \beta(\theta+\tau(t))\, x(t)\, x_t(\theta)\, d\theta +$$

$$+ \int_{-\tau(t)}^{0} \int_{\theta}^{0} \delta(\theta+\tau(t), \sigma+\tau(t))\, x_t(\theta)\, x_t(\sigma)\, d\sigma d\theta \tag{2.351}$$

The coefficients of the functional (2.351) will be obtained.
Equation (2.337) takes a form

$$\begin{bmatrix} \frac{d\beta(\xi)}{d\theta} \\ \frac{d\beta(-\xi+\tau(t))}{d\theta} \end{bmatrix} = \begin{bmatrix} \frac{-q}{T\left(1-\frac{d\tau(t)}{dt}\right)} & \frac{-k_0 p}{T\left(1-\frac{d\tau(t)}{dt}\right)^2} \\ \frac{k_0 p}{T\left(1-\frac{d\tau(t)}{dt}\right)^2} & \frac{q}{T\left(1-\frac{d\tau(t)}{dt}\right)} \end{bmatrix} \begin{bmatrix} \beta(\xi) \\ \beta(-\xi+\tau(t)) \end{bmatrix} \tag{2.352}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$.
The fundamental matrix of the differential equation (2.352) is given by the formula

$$R(\xi) = \begin{bmatrix} ch\lambda\xi - \frac{q}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)} sh\lambda\xi & -\frac{k_0 p}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)^2} sh\lambda\xi \\ \frac{k_0 p}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)^2} sh\lambda\xi & ch\lambda\xi + \frac{q}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)} sh\lambda\xi \end{bmatrix} \tag{2.353}$$

where

$$\lambda = \frac{\sqrt{q^2\left(1-\frac{d\tau(t)}{dt}\right)^2 - k_0^2 p^2}}{T\left(1-\frac{d\tau(t)}{dt}\right)^2} \tag{2.354}$$

Hence

$$\begin{bmatrix} \beta(\xi) \\ \beta(-\xi+\tau(t)) \end{bmatrix} = R(\xi)\begin{bmatrix} \beta(0) \\ \beta(\tau(t)) \end{bmatrix} \tag{2.355}$$

for $t \geq t_0$, $\xi \in [0,\tau(t)]$ where $0 \leq \tau(t) \leq r$. One needs the initial conditions of the set of differential equations (2.352) to obtain

$$\beta(\theta+\tau(t)) = \beta(\xi)\,|_{\xi=\theta+\tau(t)} \tag{2.356}$$

$$\delta(\theta+\tau(t),\sigma+\tau(t)) = -\frac{k_0 p}{T}\left(1-\frac{d\tau(t)}{dt}\right)^{-1}\beta(\sigma-\theta) \tag{2.357}$$

for $t \geq t_0$, $\theta \in [-\tau(t),0]$, $\sigma \in [\theta,0]$ where $0 \leq \tau(t) \leq r$.
The initial conditions of the differential equation (2.352) and the coefficient $\alpha(t)$ are obtained by solving the set of equations (2.343) to (2.345) which takes the form as below

$$4\frac{qk_0 p}{T^2}\alpha(t) + \left(-\frac{q}{T}\frac{d\tau(t)}{dt} - \frac{d^2\tau(t)}{dt^2}\right)\beta(0) + \left(1-\frac{1}{\frac{d\tau(t)}{dt}-1}\right)b\beta(\tau(t)) = -2bw \tag{2.358}$$

$$-\frac{2k_0 p}{T}\alpha(t) + \left(\frac{d\tau(t)}{dt}-1\right)\beta(0) = 0 \tag{2.359}$$

$$p_1\beta(0) + p_2\beta(\tau(t)) = 0 \tag{2.360}$$

where

$$p_1 = ch\frac{\lambda\tau(t)}{2} + \left(-\frac{q}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)} - \frac{k_0 p}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)^2}\right)sh\frac{\lambda\tau(t)}{2} \tag{2.361}$$

$$p_2 = -ch\frac{\lambda\tau(t)}{2} + \left(-\frac{q}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)} - \frac{k_0 p}{\lambda T\left(1-\frac{d\tau(t)}{dt}\right)^2}\right)sh\frac{\lambda\tau(t)}{2} \tag{2.362}$$

We compute the value of the performance index for initial conditions given below

$$x(0) = x_0 = 1,\ \varphi(\theta) = 0 \text{ for } \theta \in [-r,0)$$

$$J(t) = x_0^2\alpha(t) \text{ for } t \geq 0.$$

Figures show the graphs of function $J(t)$, obtained with the Matlab code, for given values of parameters $q = 1$, $T = 5$, $k_0 = 1$ and $\tau(t) = r(1 - exp(-t))$, $r = 0.5$ of system (2.348). Figure 2.11 presents the index value graph for $p = 15.11$ and Figure 2.12 for $p = 15$. The gain $p = 15.1129$ is called the critical gain. For gain greater then critical gain system (2.348) becomes unstable.

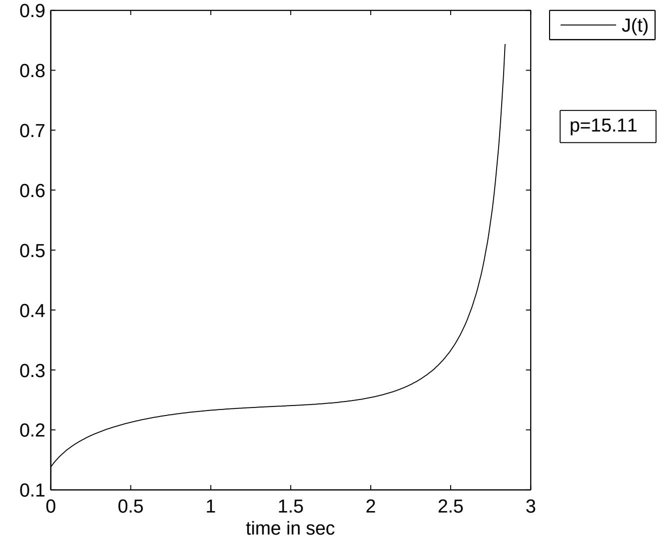

Fig. 2.11. Value of the index $J(t)$ for $p = 15.11$

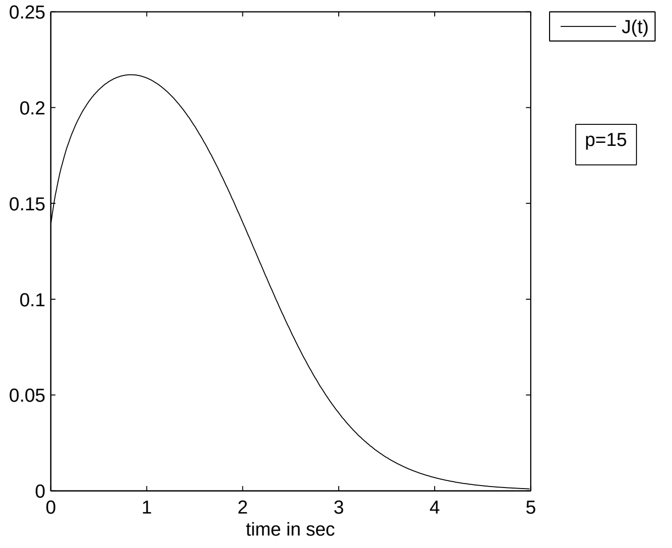

Fig. 2.12. Value of the index $J(t)$ for $p = 15$

Figure 2.13 shows the function $\beta(\xi)$ for $p = 5$ and Figure 2.14 shows the function $\beta(-\xi + \tau(t))$ for $p = 5$.

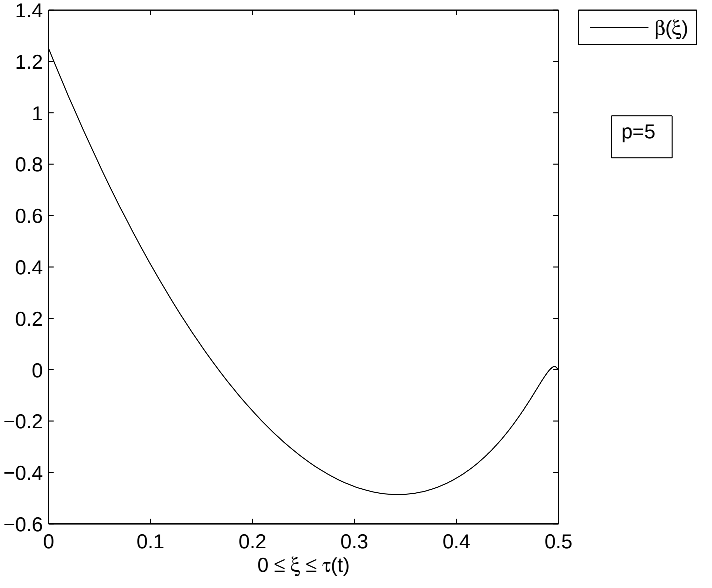

Fig. 2.13. Function $\beta(\xi)$ for $p = 5$

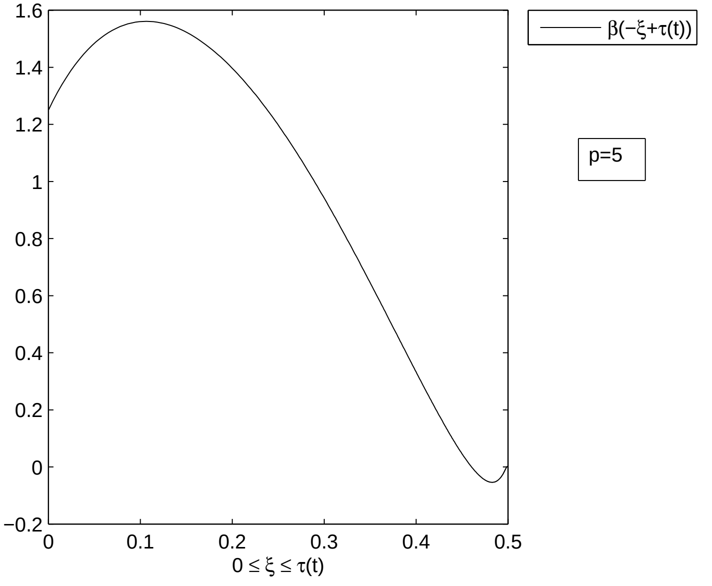

Fig. 2.14. Function $\beta(-\xi + \tau(t))$ for $p = 5$

2.5.3.2 The example. Two dimensional system

Let us consider a system described by the equation

$$\begin{cases} \begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \\ + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{t_1}(-\tau(t)) \\ x_{t_2}(-\tau(t)) \end{bmatrix} \\ \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ \begin{bmatrix} x_{t_1}(\theta) \\ x_{t_2}(\theta) \end{bmatrix} = \begin{bmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{bmatrix} \end{cases} \tag{2.363}$$

The Lyapunov functional is defined by the formula

$$V(x(t),x_t,t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} +$$

$$+ \int_{-\tau(t)}^{0} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \beta_{11}(\theta+\tau(t)) & \beta_{12}(\theta+\tau(t)) \\ \beta_{21}(\theta+\tau(t)) & \beta_{22}(\theta+\tau(t)) \end{bmatrix} \begin{bmatrix} x_{t_1}(\theta) \\ x_{t_2}(\theta) \end{bmatrix} d\theta +$$

$$+ \int_{-\tau(t)}^{0} \int_{\theta}^{0} \begin{bmatrix} x_{t_1}(\theta) & x_{t_2}(\theta) \end{bmatrix} \begin{bmatrix} \delta_{11}(\theta+\tau(t),\sigma+\tau(t)) & \delta_{12}(\theta+\tau(t),\sigma+\tau(t)) \\ \delta_{21}(\theta+\tau(t),\sigma+\tau(t)) & \delta_{22}(\theta+\tau(t),\sigma+\tau(t)) \end{bmatrix} \times$$

$$\times \begin{bmatrix} x_{t_1}(\sigma) \\ x_{t_2}(\sigma) \end{bmatrix} d\sigma d\theta \tag{2.364}$$

for $t \geq t_0$, $\theta \in [-\tau(t),0]$, $\sigma \in [\theta,0]$ where $0 \leq \tau(t) \leq r$.
The set of equations (2.337) becomes

$$\frac{d}{d\xi} \begin{bmatrix} col\,\beta(\xi) \\ col\,\beta(-\xi+\tau(t)) \end{bmatrix} = Q \begin{bmatrix} col\,\beta(\xi) \\ col\,\beta(-\xi+\tau(t)) \end{bmatrix} \tag{2.365}$$

for $\xi \in [0,\tau(t)]$, $0 \leq \tau(t) \leq r$ where

$$Q = [Q_1 Q_2] \tag{2.366}$$

$$
Q_1 = \begin{bmatrix}
\frac{a_{11}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{21}}{\frac{d\tau(t)}{dt}-1} & 0 & 0 \\
\frac{a_{12}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{22}}{\frac{d\tau(t)}{dt}-1} & 0 & 0 \\
0 & 0 & \frac{a_{11}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{21}}{\frac{d\tau(t)}{dt}-1} \\
0 & 0 & \frac{a_{12}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{22}}{\frac{d\tau(t)}{dt}-1} \\
\frac{b_{11}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{21}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & 0 & 0 \\
0 & 0 & \frac{b_{11}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{21}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} \\
\frac{b_{12}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{22}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & 0 & 0 \\
0 & 0 & \frac{b_{12}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{22}}{\left(\frac{d\tau(t)}{dt}-1\right)^2}
\end{bmatrix}
$$

$$
Q_2 = \begin{bmatrix}
\frac{b_{11}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{21}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & 0 & 0 \\
0 & 0 & \frac{b_{11}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{21}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} \\
\frac{b_{12}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{22}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & 0 & 0 \\
0 & 0 & \frac{b_{12}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} & \frac{b_{22}}{\left(\frac{d\tau(t)}{dt}-1\right)^2} \\
\frac{a_{11}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{21}}{\frac{d\tau(t)}{dt}-1} & 0 & 0 \\
\frac{a_{12}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{22}}{\frac{d\tau(t)}{dt}-1} & 0 & 0 \\
0 & 0 & \frac{a_{11}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{21}}{\frac{d\tau(t)}{dt}-1} \\
0 & 0 & \frac{a_{12}}{\frac{d\tau(t)}{dt}-1} & \frac{a_{22}}{\frac{d\tau(t)}{dt}-1}
\end{bmatrix}
$$

$$\begin{bmatrix} col\,\beta(\xi) \\ \\ col\,\beta(-\xi+\tau(t)) \end{bmatrix} = e^{Q\xi} \begin{bmatrix} col\,\beta(0) \\ \\ col\,\beta(\tau(t)) \end{bmatrix} \tag{2.367}$$

for $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$.
We introduce

$$e^{Q\frac{\tau(t)}{2}} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} & p_{48} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} & p_{58} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} & p_{68} \\ p_{71} & p_{72} & p_{73} & p_{74} & p_{75} & p_{76} & p_{77} & p_{78} \\ p_{81} & p_{82} & p_{83} & p_{84} & p_{85} & p_{86} & p_{87} & p_{88} \end{bmatrix} \tag{2.368}$$

Now we give the formulas for determination of the set of initial conditions of equation (2.365) and the matrix α.

$$\begin{bmatrix} D & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & 0_{(4,4)} \\ 0_{(4,4)} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} col\,\alpha(t) \\ col\,\beta(0) \\ col\,\beta(\tau(t)) \end{bmatrix} = \begin{bmatrix} -2b_{11}w_{11} - b_{21}w_{12} - b_{21}w_{21} \\ -b_{11}w_{12} - b_{11}w_{21} - 2b_{21}w_{22} \\ -2b_{12}w_{11} - b_{22}w_{12} - b_{22}w_{21} \\ -b_{12}w_{12} - b_{12}w_{21} - 2b_{22}w_{22} \\ 0_{(8,1)} \end{bmatrix} \tag{2.369}$$

where

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{31} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \tag{2.370}$$

$$d_{11} = 4a_{11}b_{11} + 2a_{12}b_{21} \tag{2.371}$$

$$d_{12} = d_{13} = 2a_{21}b_{11} + a_{22}b_{21} + a_{11}b_{21} \tag{2.372}$$

$$d_{14} = 2a_{21}b_{21} \tag{2.373}$$

$$d_{21} = 2a_{12}b_{11} \tag{2.374}$$

$$d_{22} = d_{23} = a_{11}b_{11} + a_{22}b_{11} + 2a_{12}b_{21} \tag{2.375}$$

$$d_{24} = 2a_{21}b_{11} + 4a_{22}b_{21} \tag{2.376}$$

$$d_{31} = 4a_{11}b_{12} + 2a_{12}b_{22} \tag{2.377}$$

$$d_{32} = d_{33} = 2a_{21}b_{12} + a_{22}b_{22} + a_{11}b_{22} \tag{2.378}$$

$$d_{34} = 2a_{21}b_{22} \tag{2.379}$$

$$d_{41} = 2a_{12}b_{12} \tag{2.380}$$

$$d_{42} = d_{43} = a_{11}b_{12} + a_{22}b_{12} + 2a_{12}b_{22} \tag{2.381}$$

$$d_{44} = 2a_{21}b_{12} + 4a_{22}b_{22} \tag{2.382}$$

$$Z_{12} = \left[Z_{12}^{1}\ Z_{12}^{2}\right] \tag{2.383}$$

$$Z_{12}^{1} = \begin{bmatrix} \frac{d\tau(t)}{dt}a_{11} - \frac{d^2\tau(t)}{dt^2} & \frac{d\tau(t)}{dt}a_{21} \\ \frac{d\tau(t)}{dt}a_{12} & \frac{d\tau(t)}{dt}a_{22} - \frac{d^2\tau(t)}{dt^2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.384}$$

$$Z_{12}^{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{d\tau(t)}{dt}a_{11} - \frac{d^2\tau(t)}{dt^2} & \frac{d\tau(t)}{dt}a_{21} \\ \frac{d\tau(t)}{dt}a_{12} & \frac{d\tau(t)}{dt}a_{22} - \frac{d^2\tau(t)}{dt^2} \end{bmatrix} \tag{2.385}$$

$$Z_{13} = \left[Z_{13}^{1}\ Z_{13}^{2}\right] \tag{2.386}$$

$$Z_{13}^{1} = \begin{bmatrix} b_{11}\left(1 - \frac{1}{\frac{d\tau(t)}{dt} - 1}\right) & -\frac{b_{21}}{\frac{d\tau(t)}{dt} - 1} \\ 0 & b_{11} \\ b_{12}\left(1 - \frac{1}{\frac{d\tau(t)}{dt} - 1}\right) & -\frac{b_{22}}{\frac{d\tau(t)}{dt} - 1} \\ 0 & b_{12} \end{bmatrix} \tag{2.387}$$

$$Z_{13}^2 = \begin{bmatrix} b_{21} & 0 \\ -\dfrac{b_{11}}{\dfrac{d\tau(t)}{dt}-1} & b_{21}\left(1-\dfrac{1}{\dfrac{d\tau(t)}{dt}-1}\right) \\ b_{22} & 0 \\ -\dfrac{b_{12}}{\dfrac{d\tau(t)}{dt}-1} & b_{22}\left(1-\dfrac{1}{\dfrac{d\tau(t)}{dt}-1}\right) \end{bmatrix} \tag{2.388}$$

$$Z_{21} = \begin{bmatrix} 2b_{11} & b_{21} & b_{21} & 0 \\ 0 & b_{11} & b_{11} & 2b_{21} \\ 2b_{12} & b_{22} & b_{22} & 0 \\ 0 & b_{12} & b_{12} & 2b_{22} \end{bmatrix} \tag{2.389}$$

$$Z_{22} = \begin{bmatrix} \dfrac{d\tau(t)}{dt}-1 & 0 & 0 & 0 \\ 0 & \dfrac{d\tau(t)}{dt}-1 & 0 & 0 \\ 0 & 0 & \dfrac{d\tau(t)}{dt}-1 & 0 \\ 0 & 0 & 0 & \dfrac{d\tau(t)}{dt}-1 \end{bmatrix} \tag{2.390}$$

$$Z_{32} = \begin{bmatrix} p_{11}-p_{51} & p_{12}-p_{52} & p_{13}-p_{53} & p_{14}-p_{54} \\ p_{21}-p_{61} & p_{22}-p_{62} & p_{23}-p_{63} & p_{24}-p_{64} \\ p_{31}-p_{71} & p_{32}-p_{72} & p_{33}-p_{73} & p_{34}-p_{74} \\ p_{41}-p_{81} & p_{42}-p_{82} & p_{43}-p_{83} & p_{44}-p_{84} \end{bmatrix} \tag{2.391}$$

$$Z_{33} = \begin{bmatrix} p_{15}-p_{55} & p_{16}-p_{56} & p_{17}-p_{57} & p_{18}-p_{58} \\ p_{25}-p_{65} & p_{26}-p_{66} & p_{27}-p_{67} & p_{28}-p_{68} \\ p_{35}-p_{75} & p_{36}-p_{76} & p_{37}-p_{77} & p_{38}-p_{78} \\ p_{45}-p_{85} & p_{46}-p_{86} & p_{47}-p_{87} & p_{48}-p_{88} \end{bmatrix} \tag{2.392}$$

Now we obtain the matrix $\delta(\theta+\tau(t),\sigma+\tau(t))$

$$\delta_{11}(\theta+\tau(t),\sigma+\tau(t)) = \frac{b_{11}}{1-\dfrac{d\tau(t)}{dt}}\beta_{11}(\theta-\sigma) + \frac{b_{21}}{1-\dfrac{d\tau(t)}{dt}}\beta_{21}(\theta-\sigma) \tag{2.393}$$

$$\delta_{12}(\theta+\tau(t),\sigma+\tau(t)) = \frac{b_{11}}{1-\dfrac{d\tau(t)}{dt}}\beta_{12}(\theta-\sigma) + \frac{b_{21}}{1-\dfrac{d\tau(t)}{dt}}\beta_{22}(\theta-\sigma) \tag{2.394}$$

$$\delta_{21}(\theta+\tau(t),\sigma+\tau(t)) = \frac{b_{12}}{1-\dfrac{d\tau(t)}{dt}}\beta_{11}(\theta-\sigma) + \frac{b_{22}}{1-\dfrac{d\tau(t)}{dt}}\beta_{21}(\theta-\sigma) \tag{2.395}$$

$$\delta_{22}(\theta+\tau(t),\sigma+\tau(t)) = \frac{b_{12}}{1-\dfrac{d\tau(t)}{dt}}\beta_{12}(\theta-\sigma) + \frac{b_{22}}{1-\dfrac{d\tau(t)}{dt}}\beta_{22}(\theta-\sigma) \tag{2.396}$$

for $t \geq t_0$, $\theta \in [-\tau(t), 0]$, $\sigma \in [\theta, 0]$ where $0 \leq \tau(t) \leq r$.
Figures 2.15–2.20 show graphs of functions $\alpha(t)$, $\beta(\xi)$, $\eta(\xi)$ obtained with the Matlab code, for given values of matrices A, B, W of system (2.363)

$$A = \begin{bmatrix} -1 & 0.6 \\ 0.5 & -2 \end{bmatrix} \; B = \begin{bmatrix} -1 & 0.4 \\ 0.1 & -1 \end{bmatrix} \; W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.397}$$

and time delay given by the function

$$\tau(t) = r\left(1 - \exp\left(-\frac{t}{T}\right)\right)$$

where $r = 0.5$, $T = 1$.

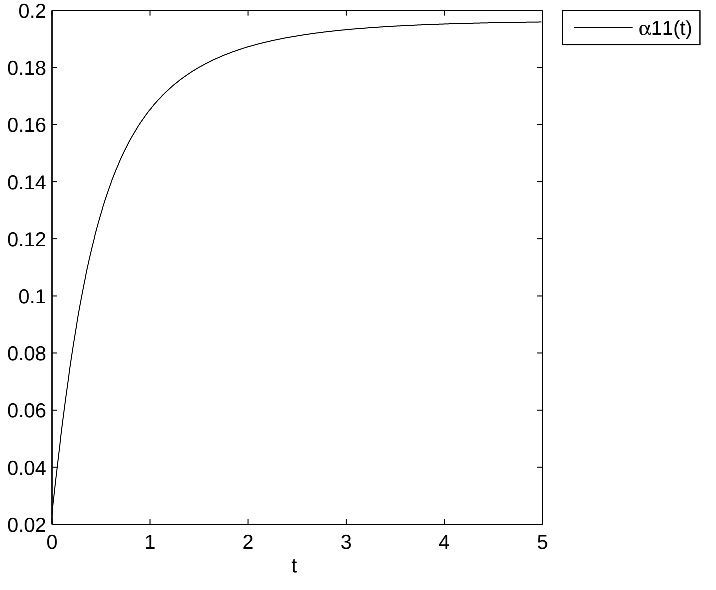

Fig. 2.15. Function $\alpha_{11}(t)$

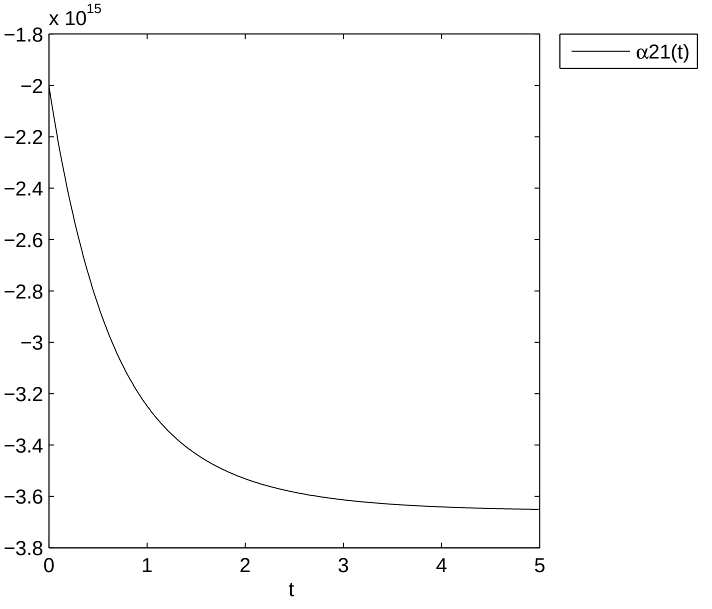

Fig. 2.16. Function $\alpha_{21}(t)$

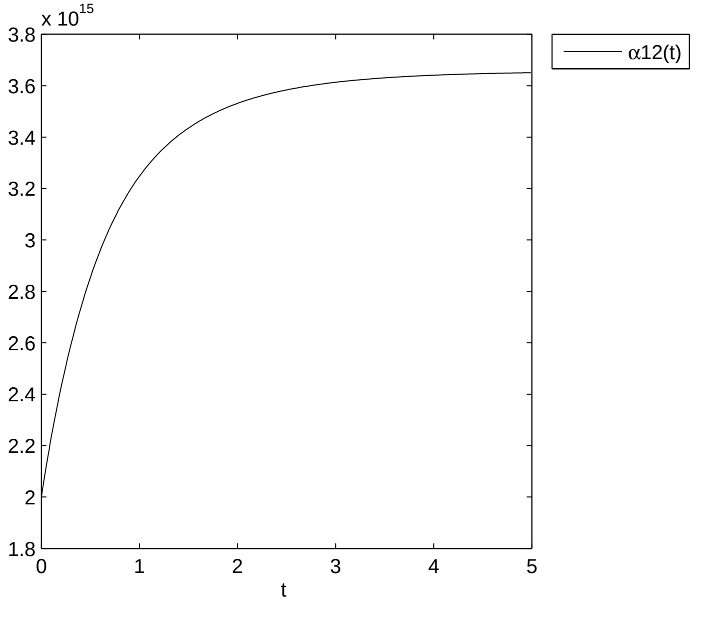

Fig. 2.17. Function $\alpha_{12}(t)$

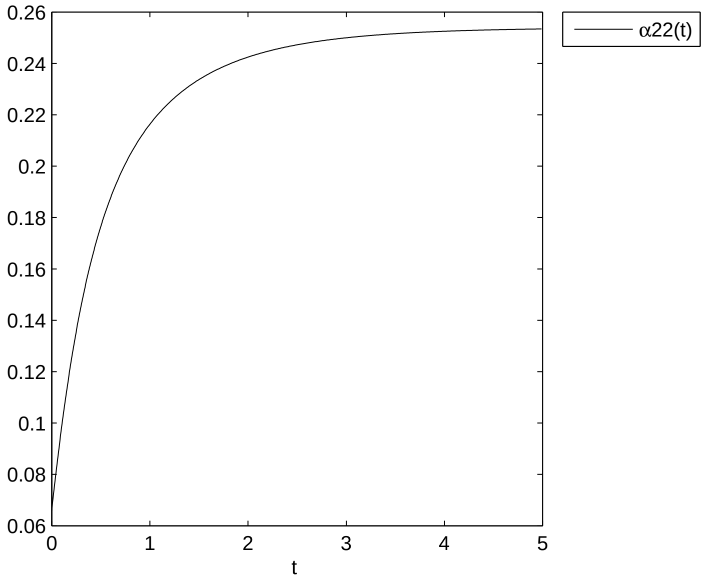

Fig. 2.18. Function $\alpha_{22}(t)$

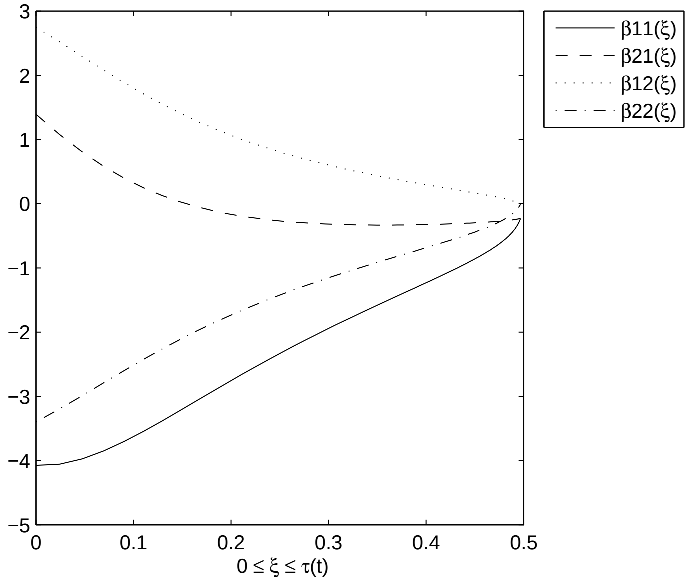

Fig. 2.19. Elements of matrix $\beta(\xi)$

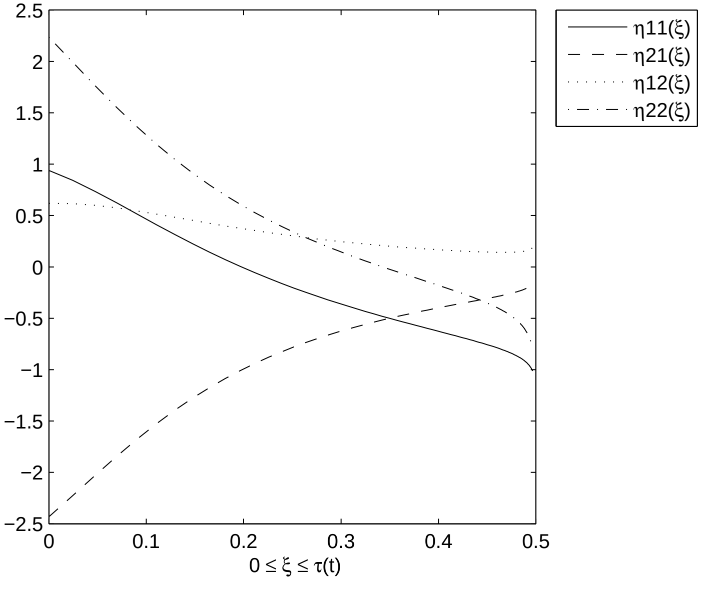

Fig. 2.20. Elements of matrix $\eta(\xi)$

3 A linear neutral system

3.1 Preliminaries

Let us consider a neutral system whose dynamics is described by the functional-differential equation [56]

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-r)}{dt} = \mathscr{L}(t, x(t), x_t) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \varphi \in W^{1,2}([-r,0), \mathbb{R}^n) \end{cases} \tag{3.1}$$

for $t \geq t_0$, $r > 0$, $x(t) \in \mathbb{R}^n$, $x_t \in W^{1,2}([-r,0), \mathbb{R}^n)$, where $W^{1,2}([-r,0), \mathbb{R}^n)$ is a space of all absolutely continuous $\mathbb{R}^n$ - valued functions with derivatives in a space of Lebesgue square integrable functions on interval $[-r,0)$ with norm $\| \varphi \|_{W^{1,2}} = \sqrt{\int_{-r}^{0} \left(\| \varphi(t) \|_{\mathbb{R}^n}^2 + \| \frac{d\varphi(t)}{dt} \|_{\mathbb{R}^n}^2 \right) dt}$
The norm of the initial value (x_0, φ) is given by the formula

$$\| (x_0, \varphi) \|_{\mathbb{R}^n \times W^{1,2}} = \sqrt{\| x_0 \|_{\mathbb{R}^n}^2 + \| \varphi \|_{W^{1,2}}^2} \tag{3.2}$$

The function $\mathscr{L}$ is linear, continuous and defined on the space $[0,\infty) \times \mathbb{R}^n \times W^{1,2}([-r,0), \mathbb{R}^n)$

$$\mathscr{L} : [0,\infty) \times \mathbb{R}^n \times W^{1,2}([-r,0), \mathbb{R}^n) \to \mathbb{R}^n$$

The solution of the functional-differential equation (3.1) with initial value (x_0, φ) for $t \geq t_0$ is an absolutely continuous function with values in $\mathbb{R}^n$ and is denoted as $x(\cdot, t_0, (x_0, \varphi))$.
We say that $x(t, t_0, (x_0, \varphi))$, for $t \geq t_0$ is a solution of system (3.1) if it satisfies the system equation (3.1) almost everywhere on $[t_0, \infty)$.
The function $x_t(t_0, (x_0, \varphi)) \in W^{1,2}([-r,0), \mathbb{R}^n)$ is a shifted restriction of function $x(\cdot, t_0, (x_0, \varphi))$ to the interval $[t-r, t)$.
The initial condition for equation (3.1) can be written in a form

$$x_{t_0}(t_0, (x_0, \varphi)) = \varphi \tag{3.3}$$

We assume that system (3.1) admits the trivial solution, i.e., the following identity holds:

$$\mathscr{L}(t, 0_{\mathbb{R}^n}, 0_{W^{1,2}}) \equiv 0$$

for $t \geq 0$.
Let $x(t, t_0, (x_0, \varphi))$ for $t \geq t_0$ be the solution of system (3.1) with initial condition (x_0, φ).

Definition 3.1. [56] *The trivial solution of system (2.1) is said to be **stable** if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that for every $(x_0, \varphi) \in \mathbb{R}^n \times W^{1,2}([-r, 0), \mathbb{R}^n)$*

$$\| (x_0, \varphi) \|_{\mathbb{R}^n \times W^{1,2}} \leq \delta(\varepsilon, t_0) \Rightarrow \| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \leq \varepsilon$$

for every $t \geq t_0$.

Definition 3.2. [56] *The trivial solution of system (2.1) is said to be **asymptotically stable** if it is stable and $\| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \to 0$ as $t - t_0 \to \infty$.*

Definition 3.3. [56] *The trivial solution of system (2.1) is said to be **exponentially stable** if there exist $\delta > 0$, $M \geq 1$ and $\sigma > 0$ such that for every $t_0 \geq 0$ and initial condition $(x_0, \varphi) \in \mathbb{R}^n \times W^{1,2}([-r, 0), \mathbb{R}^n)$, with $\| (x_0, \varphi) \|_{\mathbb{R}^n \times W^{1,2}} \leq \delta$ the following inequality holds*

$$\| x(t, t_0, (x_0, \varphi)) \|_{\mathbb{R}^n} \leq M e^{-\sigma(t - t_0)} \| (x_0, \varphi) \|_{\mathbb{R}^n \times W^{1,2}}$$

for every $t \geq t_0$.

Assumption 1. We assume that the difference $x(t, t_0, (x_0, \varphi)) - Cx(t - r, t_0, (x_0, \varphi))$ is continuous and differentiable for $t \geq t_0$, except possibly a countable number of points.

Assumption 2. We assume that there exists the right-hand-side derivative of the difference $x(t, t_0, (x_0, \varphi)) - Cx(t - r, t_0, (x_0, \varphi))$ at the point $t = t_0$.

Let $x(t, t_0, (x_0, \varphi))$ be a solution of the initial value problem (3.1) then

$$x(t, t_0, (x_0, \varphi)) = Cx(t - r, t_0, (x_0, \varphi)) + [\varphi(0) - C\varphi(-r)] + \int_{t_0}^{t} \mathscr{L}(s, x(s), x_s) ds \tag{3.4}$$

for $t \geq t_0$.
If $\theta_1 \in [-r, 0]$ is a discontinuity point of φ then according to Assumption 1 the function

$$z(t) = Cx(t - r, t_0, (x_0, \varphi)) + [\varphi(0) - C\varphi(-r)]$$

has jump points of discontinuity at $t_k = t_0 + \theta_1 + kr$, for $k \geq 1$, and the size of the jump at the points is such that $\Delta x(t_1) = C\Delta\varphi(\theta_1)$ where $\Delta x(t_1) = x(t_1 + 0) - x(t_1 - 0)$ and $\Delta x(t_{k+1}) = C\Delta x(t_k)$ for $k \geq 1$.

We obtained **the jump equation**

$$\Delta x(t_{k+1}) = C\Delta x(t_k) \tag{3.5}$$

for $k \geq 1$.
The jump equation implies

$$\Delta x(t_{k+1}) = C^k \Delta x(t_1) \tag{3.6}$$

for $k \geq 1$ and $t_k = t_0 + \theta_1 + kr$.
For a given $t \in [t_0, \infty)$ we define an integer k such that $t \in [t_0 + (k-1)r, t_0 + kr)$.
The solution (3.4) for $t \in [t_0 + (k-1)r, t_0 + kr)$ we can express in a form

$$x(t, t_0, (x_0, \varphi)) = C^k x(t - r, t_0, (x_0, \varphi)) + \sum_{j=0}^{k-1} C^j [\varphi(0) - C\varphi(-r)] +$$
$$+ \sum_{j=0}^{k-1} C^j \int_{t_0}^{t-jr} \mathscr{L}(s, x(s), x_s) ds \tag{3.7}$$

Corollary 3.1. *The system (3.1) cannot be stable if the matrix C admits an eigenvalue with magnitude greater than one.*

Indeed, if the matrix C has an eigenvalue with magnitude greater than one, then for any $\delta > 0$ there exists an initial function $(x_0, \varphi) \in \mathbb{R}^n \times W^{1,2}([-r, 0), \mathbb{R}^n)$, with $\| (x_0, \varphi) \|_{\mathbb{R}^n \times W^{1,2}} \leq \delta$, such that the corresponding solution $x(t, t_0, (x_0, \varphi))$ has a sequence of jumps, and the size of jumps tends to infinity, see (3.6) and (3.7).
The arbitrary eigenvalue of the matrix C will be denoted as $\lambda(C)$.

Definition 3.4. *The* ***spectrum*** $\sigma(C)$ *is the set of eigenvalues of matrix C, i.e. the set of complex numbers* λ *for which a matrix* $\lambda I - C$ *is not invertible.*

$$\sigma(C) = \{\lambda \in \mathbb{C} : \det(\lambda I - C) = 0\} \tag{3.8}$$

Definition 3.5. *The* ***spectral radius*** *of a matrix C is given by a form*

$$\gamma(C) = \sup\{| \lambda | : \lambda \in \sigma(C)\} \tag{3.9}$$

Definition 3.6. *The matrix C is called a* ***Schur stable matrix*** *if the eigenvalues of C lie in the interior of the unit disk of the complex plane, i.e. if the spectral radius* $\gamma(C) < 1$.

The Corollary 3.1 motivate the following assumption.
Assumption 3. In that monograph we assume that matrix C is Schur stable.

3.2 A linear neutral system with lumped delay

3.2.1 Mathematical model of a linear neutral system with lumped delay

Let us consider a linear neutral system, whose dynamics is described by the functional-differential equation [13]

$$\begin{cases} \frac{dx(t)}{dt} - \sum_{i=1}^{k} B_i \frac{dx(t-\tau_i)}{dt} = Ax(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.10}$$

for $t \geq t_0$, $\theta \in [-r, 0)$ $x(t) \in \mathbb{R}^n$, A, A_i, $B_i \in \mathbb{R}^{n\times n}$, $i = 1, ..., k$, $0 \leq \tau_1 \leq ... \leq \tau_i \leq ... \leq \tau_k = r$, $\varphi \in W^{1,2}([-r,0), \mathbb{R}^n)$, where $W^{1,2}([-r,0), \mathbb{R}^n)$ is a space of all absolutely continuous functions with derivatives in a space of Lebesgue square integrable functions on interval $[-r, 0)$ with values in $\mathbb{R}^n$. The solution of the functional-differential equation (3.10) with initial value (x_0, φ) for $t \geq t_0$ is an absolutely continuous function with values in $\mathbb{R}^n$ and is denoted as $x(\cdot, t_0, (x_0, \varphi))$.

Equation (3.10) can be written in a form

$$\begin{cases} \frac{dx(t)}{dt} - \sum_{i=1}^{k} B_i \frac{dx_t(-\tau_i)}{dt} = Ax(t) + \sum_{i=1}^{k} A_i x_t(-\tau_i) \\ x(t_0) = x_0 \\ x_{t_0} = \varphi \in W^{1,2}([-r,0), \mathbb{R}^n) \end{cases} \tag{3.11}$$

for $t \geq t_0$, where $x_t \in W^{1,2}([-r,0), \mathbb{R}^n)$ is a shifted restriction of the function $x(\cdot, t_0, (x_0, \varphi))$ to the interval $[-r, 0)$. The theorems of existence, continuous dependence and uniqueness of solutions of equation (3.11) are given in [32].

Definition 3.7. *The* ***difference equation*** *associated with (3.11) is given by a term*

$$x(t) = \sum_{i=1}^{k} B_i x_t(-\tau_i) \tag{3.12}$$

for $t \geq t_0$.

According to the Theorem 9.6.1 [40] a difference equation (3.12) for fixed rationally independent

$$0 < \tau_1 \leq ... \leq \tau_j \leq ... \leq \tau_k$$

is stable if

$$\sup \left\{ \gamma \left(\sum_{j=1}^{k} e^{i\theta_j} B_j \right) : \theta_j \in [0, 2\pi], 1 \leq j \leq k \right\} < 1 \tag{3.13}$$

where $\gamma\left(\sum_{j=1}^{k} e^{i\theta_j} B_j\right)$ is the spectral radius of a matrix $\sum_{j=1}^{k} e^{i\theta_j} B_j$.

If each B_j is a scalar then a difference equation is stable if and only if

$$\sum_{j=1}^{k} | B_j | < 1 \tag{3.14}$$

A new variable y, is defined by the formula

$$y(t) = x(t) - \sum_{i=1}^{k} B_i x_t(-\tau_i) \tag{3.15}$$

for $t \geq t_0$
Thus the equation (3.11) takes a form

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + \sum\limits_{i=1}^{k} (A_i + AB_i) x_t(-\tau_i) \\ y(t) = x(t) - \sum\limits_{i=1}^{k} B_i x_t(-\tau_i) \\ y(t_0) = x_0 - \sum\limits_{i=1}^{k} B_i \varphi(-\tau_i) \\ x_{t_0} = \varphi \end{cases} \tag{3.16}$$

Let us assume that the matrices B_i for $i = 1, ..., k$ fulfill the condition (3.13).
The state of system (3.16) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \tag{3.17}$$

for $t \geq t_0$
The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r, 0), \mathbb{R}^n) \tag{3.18}$$

The norm in the state space X is defined by a term

$$\| S(t) \|_X = \sqrt{\| y(t) \|^2_{\mathbb{R}^n} + \| x_t \|^2_{W^{1,2}}} \tag{3.19}$$

for $t \geq t_0$.
In the parametric optimization problem is used the performance index of quality, which value is given by the formula

$$J = \int_{t_0}^{\infty} y^T(t) W y(t) dt = V(y_0, \varphi) \tag{3.20}$$

where V is the Lyapunov functional defined on the state space X and W is a positive definite matrix.

3.2.2 Determination of the Lyapunov functional for a neutral system with one delay

Let us consider a system [19]

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + (A_1 + AB_1)x_t(-r) \\ y(t) = x(t) - B_1 x_t(-r) \\ y(t_0) = x_0 - B_1\varphi(-r) \\ x_{t_0} = \varphi \end{cases} \tag{3.21}$$

The state of system (3.21) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \tag{3.22}$$

for $t \geq t_0$.

The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r,0),\mathbb{R}^n) \tag{3.23}$$

On the state space X we define a quadratic functional V positive definite, differentiable, given by the formula

$$V(y(t),x_t) = y^T(t)\alpha y(t) + \int_{-r}^{0} y^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{3.24}$$

for $t \geq t_0$, where $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in C^1([-r,0],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$
$\Omega = \{(\theta,\sigma):\ \theta \in [-r,0],\ \sigma \in [\theta,0]\}$ C^1 is a space of continuous functions with continuous derivative.

In this paragraph will be given a procedure of determination of the functional (3.24) coefficients to obtain the Lyapunov functional.

The time derivative of the functional (3.24) on the trajectory of system (3.21) is computed. This time derivative is defined by the formula (2.10) which for system (3.21) takes a form

$$\frac{dV(y(t_0),\varphi)}{dt} = \limsup_{h\to 0}\frac{1}{h}\left[V\left(y(t_0+h),x_{t_0+h}\right) - V\left(y(t_0),\varphi\right)\right] \tag{3.25}$$

It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (3.24) and one substitutes in place of $dy(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dy(t)}{dt} = Ay(t) + (A_1 + AB_1)x_t(-r) \tag{3.26}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{3.27}$$

In such a manner one attains

$$\frac{dV(y(t),x_t)}{dt} = y^T(t)\left[A^T\alpha + \alpha A + \beta(0)\right]y(t) +$$
$$+ y^T(t)\left[\left(\alpha + \alpha^T\right)(A_1 + AB_1) + \beta(0)B_1 - \beta(-r)\right]x_t(-r) +$$
$$+ \int_{-r}^{0} y^T(t)\left[A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0)\right]x_t(\theta)d\theta +$$
$$+ \int_{-r}^{0} x_t^T(-r)[(A_1 + AB_1)^T\beta(\theta) + B_1^T\delta^T(\theta,0) - \delta(-r,\theta)]x_t(\theta)d\theta +$$
$$- \int_{-r}^{0}\int_{\theta}^{0} x_t^T(\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta \tag{3.28}$$

for $t \geq t_0$.
To achieve negative definiteness of that derivative we assume that

$$\frac{dV(y(t),x_t)}{dt} \equiv -y^T(t)Wy(t) \tag{3.29}$$

From relations (3.29) and (3.28) one attains the set of equations

$$A^T\alpha + \alpha A + \beta(0) = -W \tag{3.30}$$

$$\left(\alpha + \alpha^T\right)(A_1 + AB_1) + \beta(0)B_1 - \beta(-r) = 0 \tag{3.31}$$

$$A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0) = 0 \tag{3.32}$$

$$(A_1 + AB_1)^T\beta(\theta) + B_1^T\delta^T(\theta,0) - \delta(-r,\theta) = 0 \tag{3.33}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} = 0 \tag{3.34}$$

for $\theta \in [-r,0]$, $\sigma \in [\theta,0]$.
The solution of equation (3.34) is as below

$$\delta(\theta,\sigma) = f(\theta - \sigma) \tag{3.35}$$

for $\theta \in [-r,0]$, $\sigma \in [\theta,0]$, where $f \in C^1([-r,r],\mathbb{R}^{n\times n})$, C^1 is a space of continuous functions with continuous derivative.
From equation (3.32) one determines the term

$$\delta^T(\theta,0) = \frac{d\beta(\theta)}{d\theta} - A^T\beta(\theta) = f^T(\theta) \tag{3.36}$$

and one puts it into relation (3.33). After some calculations one gets

$$B_1^T \frac{d\beta(\theta)}{d\theta} + A_1^T \beta(\theta) - \delta(-r,\theta) = 0 \tag{3.37}$$

It follows from equation (3.36) that

$$\delta(-r,\theta) = f(-r-\theta) = -\frac{d\beta^T(-r-\theta)}{d\theta} - \beta^T(-r-\theta)A \tag{3.38}$$

One puts the term (3.38) into (3.37) and one obtains

$$B_1^T \frac{d\beta(\theta)}{d\theta} + \frac{d\beta^T(-r-\theta)}{d\theta} = -A_1^T \beta(\theta) - \beta^T(-r-\theta)A \tag{3.39}$$

After putting in the relation (3.39) a new variable $-r-\theta$, instead of an independent variable θ, one attains the equation

$$B_1^T \frac{d\beta(-r-\theta)}{d\theta} + \frac{d\beta^T(\theta)}{d\theta} = A_1^T \beta(-r-\theta) + \beta^T(\theta)A \tag{3.40}$$

The set of differential equations are obtained

$$\begin{cases} B_1^T \dfrac{d\beta(\theta)}{d\theta} + \dfrac{d\beta^T(-r-\theta)}{d\theta} = -A_1^T \beta(\theta) - \beta^T(-r-\theta)A \\ B_1^T \dfrac{d\beta(-r-\theta)}{d\theta} + \dfrac{d\beta^T(\theta)}{d\theta} = A_1^T \beta(-r-\theta) + \beta^T(\theta)A \end{cases} \tag{3.41}$$

The new function is given

$$\kappa(\theta) = \beta(-r-\theta) \tag{3.42}$$

The set of equations (3.41) takes a form

$$\begin{cases} B_1^T \dfrac{d\beta(\theta)}{d\theta} + \dfrac{d\kappa^T(\theta)}{d\theta} = -A_1^T \beta(\theta) - \kappa^T(\theta)A \\ B_1^T \dfrac{d\kappa(\theta)}{d\theta} + \dfrac{d\beta^T(\theta)}{d\theta} = A_1^T \kappa(\theta) + \beta^T(\theta)A \end{cases} \tag{3.43}$$

The set of equations (3.43) can be written in the form

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} - B_1^T \dfrac{d\beta^T(\theta)}{d\theta} B_1 = A_1^T \beta(\theta) B_1 + A^T \beta(\theta) + \kappa^T(\theta)(AB_1 + A_1) \\ \dfrac{d\kappa(\theta)}{d\theta} - B_1^T \dfrac{d\kappa^T(\theta)}{d\theta} B_1 = -\beta^T(\theta)(A_1 + AB_1) - A^T \kappa(\theta) - A_1^T \kappa(\theta) B_1 \end{cases} \tag{3.44}$$

To obtain the solution of equations (3.44) one needs the initial values $\beta(-r)$ and $\kappa(-r)$.

Equation (3.42) implies that

$$\kappa(-r) = \beta(0) \tag{3.45}$$

$$\beta(\theta)\mid_{\theta=-\frac{r}{2}} = \kappa(\theta)\mid_{\theta=-\frac{r}{2}} \tag{3.46}$$

Equations (3.30) and (3.31) take a form

$$A^T\alpha + \alpha A + \kappa(-r) = -W \tag{3.47}$$

$$\left(\alpha+\alpha^T\right)(A_1 + AB_1) + \kappa(-r)B_1 - \beta(-r) = 0 \tag{3.48}$$

The set of algebraic equations (3.46) to (3.48) enables to obtain the matrix α and the initial conditions of the ordinary differential equations (3.44).

3.2.3 The example. Inertial system with delay and a PD controller

Let us consider a first order inertial system with delay described by the equation [19]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{q}{T}x(t) + \dfrac{k_0}{T}u(t-r) \\ x(t_0) = x_o \\ x(t_0+\theta) = 0 \\ u(t) = -px(t) - T_d\dfrac{dx(t)}{dt} \end{cases} \tag{3.49}$$

$t \geq t_0$, $x(t) \in \mathbb{R}$, $\theta \in [-r, 0)$, p, k_0, T, T_d, q, $x_0 \in \mathbb{R}$, $r \geq 0$.
The parameter k_0 is a gain of a plant, p is a proportional gain, T_d is a derivative gain, T is a system time constant, x_0 is an initial state of system. In the case $q = 1$ an equation (3.49) describes a static object and in the case $q = 0$ an equation (3.49) describes an astatic object. One can reshape an equation (3.49) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} + \dfrac{k_0T_d}{T}\dfrac{dx(t-r)}{dt} = -\dfrac{q}{T}x(t) - \dfrac{k_0p}{T}x(t-r) \\ x(t_0) = x_o \\ x(\theta) = 0 \end{cases} \tag{3.50}$$

for $t \geq t_0$ and $\theta \in [-r, 0)$.
It is assumed that the element k_0T_d/T satisfies the condition (3.14), whose takes a form

$$\left|\frac{k_0T_d}{T}\right| < 1 \tag{3.51}$$

The Lyapunov functional V is defined by the formula

$$V(y(t),x(t+\cdot)) = \alpha y^2(t) + \int_{-r}^{0} \beta(\theta)y(t)x(t+\theta)d\theta +$$

$$+ \int_{-r}^{0}\int_{\theta}^{0} \delta(\theta,\sigma)x(t+\theta)x(t+\sigma)d\sigma d\theta \tag{3.52}$$

where

$$y(t) = x(t) + \frac{k_0 T_d}{T} x(t-r) \tag{3.53}$$

In a parametric optimization problem is used the integral quadratic performance index of quality

$$J = \int_{t_0}^{\infty} wy^2(t)dt = V(y(t_0),\varphi) \tag{3.54}$$

The set of equations (3.44) takes a form

$$\begin{bmatrix} \dfrac{d\beta(\theta)}{d\theta} \\ \dfrac{d\kappa(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} p_1 & -p_2 \\ p_2 & -p_1 \end{bmatrix} \begin{bmatrix} \beta(\theta) \\ \kappa(\theta) \end{bmatrix} \tag{3.55}$$

where

$$p_1 = \frac{k_0^2 pT - qT}{T^2 - k_0^2 T_d^2} \tag{3.56}$$

$$p_2 = \frac{k_0 pT - qk_0 T_d}{T^2 - k_0^2 T_d^2} \tag{3.57}$$

The fundamental matrix of solutions of equation (3.55) is given by the term

$$R(\theta) = \begin{bmatrix} \cosh(\lambda\theta) + \dfrac{p_1}{\lambda}\sinh(\lambda\theta) & -\dfrac{p_2}{\lambda}\sinh(\lambda\theta) \\ \dfrac{p_2}{\lambda}\sinh(\lambda\theta) & \cosh(\lambda\theta) - \dfrac{p_1}{\lambda}\sinh(\lambda\theta) \end{bmatrix} \tag{3.58}$$

where

$$\lambda = \sqrt{p_1^2 - p_2^2} = \sqrt{\frac{q^2 - k_0^2 p^2}{T^2 - k_0^2 T_d^2}} \tag{3.59}$$

The solution of the set of equations (3.58) is given by the formula

$$\beta(\theta) = [\cosh(\lambda\theta + r) + \frac{p_1}{\lambda}\sinh(\lambda\theta + r)]\beta(-r) - \frac{p_2}{\lambda}\sinh(\lambda\theta + r)\kappa(-r) \tag{3.60}$$

$$\kappa(\theta) = \frac{p_2}{\lambda}\sinh(\lambda\theta + r)\beta(-r) + [\cosh(\lambda\theta + r) - \frac{p_1}{\lambda}\sinh(\lambda\theta + r)]\kappa(-r) \tag{3.61}$$

The initial conditions of equation (3.58) and the coefficient α are obtained from the set of algebraic equations (3.46)–(3.48) which takes a form

$$\left[\cosh\left(\frac{\lambda r}{2}\right)+\frac{p_1-p_2}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)\right]\beta(-r)+$$

$$+\left[-\cosh\left(\frac{\lambda r}{2}\right)+\frac{p_1-p_2}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)\right]\kappa(-r)=0 \tag{3.62}$$

$$-\frac{2q}{T}\alpha+\kappa(-r)=-w \tag{3.63}$$

$$2\frac{qk_0T_d-k_0pT}{T^2}\alpha-\frac{k_0T_d}{T}\kappa(-r)-\beta(-r)=0 \tag{3.64}$$

From equation (3.63) one can determine a term $\kappa(-r)$ and substitute it into (3.64) and (3.62).

$$\kappa(-r)=-w+\frac{2q}{T}\alpha \tag{3.65}$$

From equation (3.64) one determines a term $\beta(-r)$

$$\beta(-r)=-2\frac{k_0p}{T}\alpha+\frac{k_0T_d}{T}w \tag{3.66}$$

One puts the terms (3.65) and (3.66) into (3.62) and one obtains a parameter α

$$\alpha=\frac{w}{2}\frac{\frac{(T-k_0T_d)(p_1-p_2)}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)-(T+k_0T_d)\cosh\left(\frac{\lambda r}{2}\right)}{\frac{(q-k_0p)(p_1-p_2)}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)-(q+k_0p)\cosh\left(\frac{\lambda r}{2}\right)} \tag{3.67}$$

The coefficient $\delta(\theta,\sigma)$ is obtained from equations (3.35), (3.36) and (3.55)

$$\delta(\theta,\sigma)=(p_1+\frac{q}{T})\beta(\theta-\sigma)-p_2\kappa(\theta-\sigma) \tag{3.68}$$

We compute the value of the performance index (3.54) for initial function Φ given by a term

$$\varphi(\theta)=\begin{cases}x_0 & \text{for } \theta=0\\ 0 & \text{for } \theta\in[-r,0)\end{cases} \tag{3.69}$$

After calculations one obtains

$$J=\frac{x_0^2w}{2}\frac{\frac{(T-k_0T_d)(p_1-p_2)}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)-(T+k_0T_d)\cosh\left(\frac{\lambda r}{2}\right)}{\frac{(q-k_0p)(p_1-p_2)}{\lambda}\sinh\left(\frac{\lambda r}{2}\right)-(q+k_0p)\cosh\left(\frac{\lambda r}{2}\right)} \tag{3.70}$$

We search for an optimal parameters of a PD-controller which minimize the index (3.70). Optimization results are given in Table 3.1. These results are obtained for $x_0 = 1$, $w = 1$, $q = 1$, $T = 5$, and $k_0 = 1$.

Table 3.1
Optimization results

Delay r	Optimal p	Optimal T_d	Index value
1.0	5.0838	2.3664	1.0245
1.5	3.3438	2.2797	1.3567
2.0	2.4745	2.1945	1.6096
2.5	1.9532	2.1110	1.8035
3.0	1.6055	2.0290	1.9528
3.5	1.3569	1.9486	2.0685
4.0	1.1699	1.8698	2.1586

3.3 The Lyapunov functional for a neutral system with both lumped and distributed time delay

3.3.1 Mathematical model of a linear neutral system with both lumped and distributed time delay

Let us consider a linear neutral system with both lumped and distributed time delay, which dynamics is described by the functional-differential equation [21]

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-r)}{dt} = Ax(t) + Bx(t-r) + \displaystyle\int_{-r}^{0} Gx(t+\theta)d\theta \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.71}$$

for $t \geq t_0, r \geq 0,\ A, B, C, G \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $\theta \in [-r,0)$, $\varphi \in W^{1,2}([-r,0),\mathbb{R}^n)$ where $W^{1,2}([-r,0),\mathbb{R}^n)$ is a space of all absolutely continuous functions $[-r,0) \to \mathbb{R}^n$ with derivatives in $L^2([-r,0),\mathbb{R}^n)$ a space of Lebesgue square integrable functions on an interval $[-r,0)$ with values in $\mathbb{R}^n$.

The solution of the functional-differential equation (3.71) with initial value (x_0, φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$ and is denoted as $x(\cdot, t_0, (x_0, \varphi))$.
Equation (3.71) can be written in the form

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx_t(-r)}{dt} = Ax(t) + Bx_t(-r) + \displaystyle\int_{-r}^{0} Gx_t(\theta)d\theta \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \varphi \in W^{1,2}([-r,0), \mathbb{R}^n) \end{cases} \tag{3.72}$$

for $t \geq t_0$, where $x_t \in W^{1,2}([-r,0), \mathbb{R}^n)$ is a shifted restriction of $x(\cdot, t_0, (x_0, \varphi))$ to an interval $[t-r, t)$.
The theorems of existence, continuous dependence and uniqueness of solutions of equation (3.72) are given in [34].

Definition 3.8. *The* ***difference equation*** *associated with (3.71) and (3.72) is given by a term*

$$x(t) = Cx(t-r) \tag{3.73}$$

for $t \geq t_0$

The eigenvalues of neutral equation (3.72) for large modulus are asymptotically equal to the eigenvalues of the difference equation (3.73). The stability of the difference equation (3.73) is the necessary condition of the stability of the neutral equation (3.72).
According to the Theorem 9.6.1 [40] the difference equation (3.73) is stable when the spectral radius $\gamma(C)$ of the matrix C fulfills the condition

$$\gamma(C) < 1 \tag{3.74}$$

We assume that the matrix C is not singular and fulfills the condition (3.74).
We introduce a new function y, defined by a term

$$y(t) = x(t) - Cx_t(-r) \tag{3.75}$$

for $t \geq t_0$.
Thus the equation (3.72) takes a form

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + (AC+B)x_t(-r) + \displaystyle\int_{-r}^{0} Gx_t(\theta)d\theta \\ y(t) = x(t) - Cx_t(-r) \\ x_{t_0} = \varphi \in W^{1,2}([-r,0), \mathbb{R}^n) \\ y(t_0) = y_0 \end{cases} \tag{3.76}$$

for $t \geq t_0$ where $y_0 = x_0 - C\varphi(-r)$.

The state of system (3.76) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \tag{3.77}$$

for $t \geq t_0$.
The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r,0),\mathbb{R}^n) \tag{3.78}$$

In the parametric optimization problem is used the performance index of quality, which value is given by the term

$$J = \int_{t_0}^{\infty} y^T(t)Wy(t)dt = V(y_0,\varphi) \tag{3.79}$$

where V is the Lyapunov functional defined on the state space X and W is a positive definite matrix. The controllability of systems with time delay is presented in [69].

3.3.2 Determination of the Lyapunov functional coefficients

On the state space X we define a quadratic functional V positive definite, differentiable, given by the formula [21]

$$V(y(t),x_t) = y^T(t)\alpha y(t) + \int_{-r}^{0} y^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0}\int_{-r}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{3.80}$$

$\alpha = \alpha^T \in \mathbb{R}^{n\times n}$; $\beta \in C^1([-r,0],\mathbb{R}^{n\times n})$; $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$;
$\Omega = \{(\theta,\sigma) : \theta \in [-r,0],\ \sigma \in [-r,0]\}$;
C^1 is a space of continuous functions with a continuous derivative.
In this paragraph will be given a procedure of determination of the functional (3.80) coefficients to obtain the Lyapunov functional. The time derivative of the functional (3.80) on the trajectory of system (3.76) is computed. This time derivative is defined by the formula (2.10) which for system (3.76) takes a form

$$\frac{dV(y(t_0),\varphi)}{dt} = \limsup_{h\to 0}\frac{1}{h}\left[V\left(y(t_0+h),x_{t_0+h}\right) - V(y(t_0),\varphi)\right] \tag{3.81}$$

It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (3.80) and one substitutes in place of $dy(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dy(t)}{dt} = Ay(t) + (AC+B)x_t(-r) + \int_{-r}^{0} Gx_t(\theta)d\theta \tag{3.82}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{3.83}$$

In such a manner one attains

$$\begin{aligned}
\frac{dV(y(t),x_t)}{dt} &= y^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2}\right]y(t)+ \\
&+ y^T(t)\left[2\alpha(AC+B) + \beta(0)C - \beta(-r)\right]x_t(-r)+ \\
&+ \int_{-r}^{0} y^T(t)\left[2\alpha G + A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta(0,\theta) + \delta^T(\theta,0)\right]x_t(\theta)d\theta+ \\
&+ \int_{-r}^{0} x_t^T(-r)[(AC+B)^T\beta(\theta) + C^T\delta(0,\theta) - \delta(-r,\theta) + C^T\delta^T(\theta,0) - \delta^T(\theta,-r)]x_t(\theta)d\theta+ \\
&+ \int_{-r}^{0}\int_{-r}^{0} x_t^T(\theta)\left[G^T\beta(\sigma) - \frac{\partial\delta(\theta,\sigma)}{\partial\theta} - \frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta
\end{aligned} \tag{3.84}$$

To achieve negative definiteness of that derivative we assume that

$$\frac{dV(y(t),x_t)}{dt} \equiv -y^T(t)Wy(t) \tag{3.85}$$

From relations (3.85) and (3.84) one attains the set of equations

$$A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} = -W \tag{3.86}$$

$$2\alpha(AC+B) + \beta(0)C - \beta(-r) = 0 \tag{3.87}$$

$$2\alpha G + A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta(0,\theta) + \delta^T(\theta,0) = 0 \tag{3.88}$$

$$(AC+B)^T\beta(\theta) + C^T\delta(0,\theta) + C^T\delta^T(\theta,0) - \delta(-r,\theta) - \delta^T(\theta,-r) = 0 \tag{3.89}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} = G^T\beta(\sigma) \tag{3.90}$$

for $\theta,\ \sigma \in [-r,0]$.
Let us consider a solution of equation (3.90) as below

$$\delta(\theta,\sigma) = f(\theta-\sigma) + f^T(\sigma-\theta) + \int_{0}^{\sigma} G^T\beta(\xi)d\xi \tag{3.91}$$

where $f \in C^1([-r,r])$.

From equation (3.91) it attains

$$\delta(0,\theta)+\delta^T(\theta,0)=2f^T(\theta)+2f(-\theta)+\int\limits_0^\theta G^T\beta(\xi)d\xi \tag{3.92}$$

and

$$\delta(-r,\theta)+\delta^T(\theta,-r)=2f(-r-\theta)+2f^T(\theta+r)+\int\limits_0^\theta G^T\beta(\xi)d\xi+\int\limits_0^{-r}\beta^T(\xi)Gd\xi \tag{3.93}$$

We put a term (3.92) into equation (3.88) and we obtain the formula

$$2\alpha G+A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+2f^T(\theta)+2f(-\theta)+\int\limits_0^\theta G^T\beta(\xi)d\xi=0 \tag{3.94}$$

Now we put the terms (3.92) and (3.93) into equation (3.89) and we get a relationship

$$\begin{aligned}&\left(C^TA^T+B^T\right)\beta(\theta)+C^T\left(2f^T(\theta)+2f(-\theta)\right)-2f(-r-\theta)-2f^T(\theta+r)+\\&+(C^T-I)\cdot\int\limits_0^\theta G^T\beta(\xi)d\xi-\int\limits_0^{-r}\beta^T(\xi)Gd\xi=0\end{aligned} \tag{3.95}$$

From equation (3.94) we attain the term

$$2f(\theta)+2f^T(-\theta)=\frac{d\beta^T(\theta)}{d\theta}-\beta^T(\theta)A-2G^T\alpha-\int\limits_0^\theta\beta^T(\xi)Gd\xi \tag{3.96}$$

and the term

$$\begin{aligned}2f(-\theta-r)+2f^T(\theta+r)=&-\frac{d\beta^T(-\theta-r)}{d\theta}-\beta^T(-\theta-r)+\\&-2G^T\alpha-\int\limits_0^{-\theta-r}\beta^T(\xi)Gd\xi\end{aligned} \tag{3.97}$$

We put the terms (3.96) and (3.97) into equation (3.95) and after some computations we obtain the formula

$$\begin{aligned}C^T\frac{d\beta(\theta)}{d\theta}+\frac{d\beta^T(-\theta-r)}{d\theta}=&-B^T\beta(\theta)-\beta^T(-\theta-r)A+\int\limits_0^\theta G^T\beta(\xi)d\xi+\\&+\int\limits_0^\theta\beta^T(-\xi-r)Gd\xi+2C^T\alpha G-2G^T\alpha\end{aligned} \tag{3.98}$$

In computations we used a relationship

$$-\int_{-r}^{-\theta-r} \beta^T(\xi)Gd\xi = \int_0^{\theta} \beta^T(-\xi-r)Gd\xi$$

We introduce a substitution

$$\frac{d\beta(\theta)}{d\theta} = \vartheta(\theta) \tag{3.99}$$

for $\theta \in [-r,0]$
We compute a derivative of a term $\beta^T(-\theta-r)$

$$\frac{d\beta^T(-\theta-r)}{d\theta} = -\vartheta^T(-\theta-r) \tag{3.100}$$

for $\theta \in [-r,0]$.
We can write equation (3.98) in a form

$$C^T\vartheta(\theta) - \vartheta^T(-\theta-r) = -B^T\beta(\theta) - \beta^T(-\theta-r)A + \int_0^{\theta} G^T\beta(\xi)d\xi +$$

$$+\int_0^{\theta} \beta^T(-\xi-r)Gd\xi + 2C^T\alpha G - 2G^T\alpha \tag{3.101}$$

Taking into account the formulas (3.99) and (3.100) we calculate a derivative of both sides of equation (3.101)

$$C^T\frac{d\vartheta(\theta)}{d\theta} - \frac{d\vartheta^T(-\theta-r)}{d\theta} = -B^T\vartheta(\theta) + \vartheta^T(-\theta-r)A +$$

$$+G^T\beta(\theta) + \beta^T(-\theta-r)G \tag{3.102}$$

for $\theta \in [-r,0]$.
We transpose both sides of equation (3.102) and then we change a variable putting $\theta = -\xi - r$ and $d\theta = -d\xi$. In this way we obtain

$$\frac{d\vartheta(\xi)}{d\xi} - \frac{d\vartheta^T(-\xi-r)}{d\xi}C = -\vartheta^T(-\xi-r)B + A^T\vartheta(\xi) +$$

$$+\beta^T(-\xi-r)G + G^T\beta(\xi) \tag{3.103}$$

for $\xi \in [-r,0]$. The sense of the formula (3.103) does not depend on the notation of the variable, so we can use symbol θ instead of ξ.

We introduce new functions

$$\kappa(\theta) = \beta^T(-\theta - r) \tag{3.104}$$

and

$$\eta(\theta) = \vartheta^T(-\theta - r) \tag{3.105}$$

for $\theta \in [-r, 0]$.
Formula (3.101) takes a form

$$C^T\vartheta(\theta) - \eta(\theta) = -B^T\beta(\theta) - \kappa(\theta)A + \int_0^\theta G^T\beta(\xi)d\xi +$$

$$+ \int_0^\theta \kappa(\xi)Gd\xi + 2C^T\alpha G - 2G^T\alpha \tag{3.106}$$

for $\theta \in [-r, 0]$.
From equations (3.104), (3.100) and (3.105) it results that

$$\frac{d\kappa(\theta)}{d\theta} = -\eta(\theta) \tag{3.107}$$

for $\theta \in [-r, 0]$.
Using the definitions (3.104) and (3.105) we can rewrite the relationships (3.102) and (3.103) in a form

$$C^T\frac{d\vartheta(\theta)}{d\theta} - \frac{d\eta(\theta)}{d\theta} = -B^T\vartheta(\theta) + \eta(\theta)A + G^T\beta(\theta) + \kappa(\theta)G \tag{3.108}$$

$$\frac{d\vartheta(\theta)}{d\theta} - \frac{d\eta(\theta)}{d\theta}C = -\eta(\theta)B + A^T\vartheta(\theta) + \kappa(\theta)G + G^T\beta(\theta) \tag{3.109}$$

for $\theta \in [-r, 0]$.
We reshape a set of equations (3.108) and (3.109) then we add to them the equations (3.99) and (3.107). In this way we obtain the differential equations set

$$\begin{cases} \dfrac{d\beta(\theta)}{d\theta} = \vartheta(\theta) \\ \dfrac{d\kappa(\theta)}{d\theta} = -\eta(\theta) \\ \dfrac{d\vartheta(\theta)}{d\theta} - C^T\dfrac{d\vartheta(\theta)}{d\theta}C = G^T\beta(\theta) - G^T\beta(\theta)C + \kappa(\theta)G(I-C) + A^T\vartheta(\theta) + \\ \qquad\qquad +B^T\vartheta(\theta)C - \eta(\theta)(B + AC) \\ \dfrac{d\eta(\theta)}{d\theta} - C^T\dfrac{d\eta(\theta)}{d\theta}C = -\left(I - C^T\right)G^T\beta(\theta) - \kappa(\theta)G + C^T\kappa(\theta)G + \\ \qquad\qquad + \left(B^T + C^TA^T\right)\vartheta(\theta) - \eta(\theta)A - C^T\eta(\theta)B \end{cases} \tag{3.110}$$

for $\theta \in [-r, 0]$ with initial conditions $\beta(-r)$, $\kappa(-r)$, $\vartheta(-r)$, $\eta(-r)$.

From formulas (3.104) and (3.105) it implies that the solution of equation (3.110) satisfies the relationships

$$\kappa(\theta)\mid_{\theta=-\frac{r}{2}}=\beta^T(\theta)\mid_{\theta=-\frac{r}{2}} \tag{3.111}$$

$$\eta(\theta)\mid_{\theta=-\frac{r}{2}}=\vartheta^T(\theta)\mid_{\theta=-\frac{r}{2}} \tag{3.112}$$

We determine a value of the initial conditions of system (3.110) to obtain a solution of the set of differential equations (3.110) on the interval $[-r,0]$.
From formulas (3.104) and (3.105) it implies that there exist the connections between initial conditions

$$\beta(0)=\kappa^T(-r) \qquad \kappa(0)=\beta^T(-r) \qquad \vartheta(0)=\eta^T(-r) \qquad \eta(0)=\vartheta^T(-r) \tag{3.113}$$

We calculate a value of a formula (3.106) for $\theta=0$. Taking into account the relationships (3.113) after transposition we obtain

$$\eta(-r)C-\vartheta(-r)+\kappa(-r)B+A^T\beta(-r)-2G^T\alpha C+2\alpha G=0 \tag{3.114}$$

Taking into consideration the conditions (3.113) we rewrite equations (3.86) and (3.87) into a form

$$A^T\alpha+\alpha A+\frac{\kappa(-r)+\kappa^T(-r)}{2}=-W \tag{3.115}$$

$$2\alpha(AC+B)+\kappa^T(-r)C-\beta(-r)=0 \tag{3.116}$$

The set of equations (3.114) - (3.116) and the terms (3.111) and (3.112) enable us to compute the initial conditions of the differential equations (3.110) and the matrix α. This equations composition constitutes the algebraic equations set with unknown $\beta(-r)$, $\kappa(-r)$, $\vartheta(-r)$, $\eta(-r)$, α.
Taking into account a term (3.99) we can write a formula (3.96) in a form

$$f(\theta)+f^T(-\theta)=\frac{1}{2}\vartheta^T(\theta)-\frac{1}{2}\beta^T(\theta)A-G^T\alpha-\frac{1}{2}\int_0^\theta\beta^T(\xi)Gd\xi \tag{3.117}$$

According to a formula (3.91) and (3.117) we attain a term

$$\delta(\theta,\sigma)=\frac{1}{2}\vartheta^T(\theta-\sigma)-\frac{1}{2}\beta^T(\theta-\sigma)A+$$

$$-\frac{1}{2}\int_0^{\theta-\sigma}\beta^T(\xi)Gd\xi+\int_0^\sigma G^T\beta(\xi)d\xi-G^T\alpha \tag{3.118}$$

In this way we obtained all the Lyapunov functional coefficients.

3.3.3 The example

Let us consider a system described by the equation [21]

$$\begin{cases} \dfrac{dx(t)}{dt} - c\dfrac{dx(t-r)}{dt} = ax(t) + bx(t-r) + \int\limits_{-r}^{0} gx(t+\theta)d\theta \\ x(t_0) = x_0 \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.119}$$

for $t \geq t_0$, $\theta \in [-r,0)$, $x(t) \in \mathbb{R}$, $\varphi \in W^{1,2}([-r,0),\mathbb{R})$, a, b, c, $g \in \mathbb{R}$, $c \neq 0$, $\mid c \mid < 1$, $r \geq 0$.
We introduce a new variable

$$y(t) = x(t) - cx(t-r) \tag{3.120}$$

for $t \geq t_0$.
Formula (3.119) takes a form

$$\begin{cases} \dfrac{dy(t)}{dt} = ay(t) + (ac+b)x(t-r) + \int\limits_{-r}^{0} gx(t+\theta)d\theta \\ y(t) = x(t) - cx(t-r) \\ y(t_0) = x_0 - c\varphi(-r) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.121}$$

for $t \geq t_0$, $\theta \in [-r,0)$, $y(t) \in \mathbb{R}$, $\varphi \in W^{1,2}([-r,0),\mathbb{R})$, a, b, c, $g \in \mathbb{R}$, $c \neq 0$, $\mid c \mid < 1$, $r \geq 0$.
The Lyapunov functional is defined by the formula

$$V(y(t),x(t+\cdot)) = \alpha y^2(t) + \int\limits_{-r}^{0} y(t)\beta(\theta)x(t+\theta)d\theta +$$
$$+ \int\limits_{-r}^{0}\int\limits_{-r}^{0} \delta(\theta,\sigma)x(t+\theta)x(t+\sigma)d\sigma d\theta \tag{3.122}$$

We write the set of differential equations (3.110) for system (3.121)

$$\begin{bmatrix} \dfrac{d\beta(\theta)}{d\theta} \\ \dfrac{d\kappa(\theta)}{d\theta} \\ \dfrac{d\vartheta(\theta)}{d\theta} \\ \dfrac{d\eta(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \dfrac{g}{1+c} & \dfrac{g}{1+c} & \dfrac{a+bc}{1-c^2} & -\dfrac{b+ac}{1-c^2} \\ -\dfrac{g}{1+c} & -\dfrac{g}{1+c} & \dfrac{b+ac}{1-c^2} & -\dfrac{a+bc}{1-c^2} \end{bmatrix} \begin{bmatrix} \beta(\theta) \\ \kappa(\theta) \\ \vartheta(\theta) \\ \eta(\theta) \end{bmatrix} \tag{3.123}$$

for $\theta \in [-r,0]$.

The solution of the differential equations (3.123) system is given by a term

$$\begin{bmatrix} \beta(\theta) \\ \kappa(\theta) \\ \vartheta(\theta) \\ \eta(\theta) \end{bmatrix} = \Psi(\theta + r) \begin{bmatrix} \beta(-r) \\ \kappa(-r) \\ \vartheta(-r) \\ \eta(-r) \end{bmatrix} \tag{3.124}$$

for $\theta \in [-r, 0]$, where $\Psi(\theta)$ is a fundamental matrix of system (3.123). The coefficient α and initial conditions of system (3.123) one obtains by solving the algebraic equations set

$$\begin{cases} 2a\alpha + \kappa(-r) = -w \\ 2(ac+b)\alpha - \beta(-r) + c\kappa(-r) = 0 \\ 2g(1-c)\alpha + a\beta(-r) + b\kappa(-r) - \vartheta(-r) + c\eta(-r) = 0 \\ \beta(\theta) \mid_{\theta=-\frac{r}{2}} = \kappa(\theta) \mid_{\theta=-\frac{r}{2}} \\ \vartheta(\theta) \mid_{\theta=-\frac{r}{2}} = \eta(\theta) \mid_{\theta=-\frac{r}{2}} \end{cases} \tag{3.125}$$

where w is a positive real number.
Having solution of equation (3.123) we can obtain a coefficient $\delta(\theta,\sigma)$

$$\delta(\theta,\sigma) = \frac{1}{2}\vartheta(\theta-\sigma) - \frac{1}{2}a\beta(\theta-\sigma) - g\alpha - \frac{1}{2}\int_0^{\theta-\sigma} g\beta(\xi)d\xi + \int_0^{\sigma} g\beta(\xi)d\xi \tag{3.126}$$

Figure 3.1 shows the functions $\beta(\theta)$,$\kappa(\theta)$,$\vartheta(\theta)$,$\eta(\theta)$ graphs and the α value attained by means of the Matlab code for given parameters a, b, c, g, w of system (3.119).

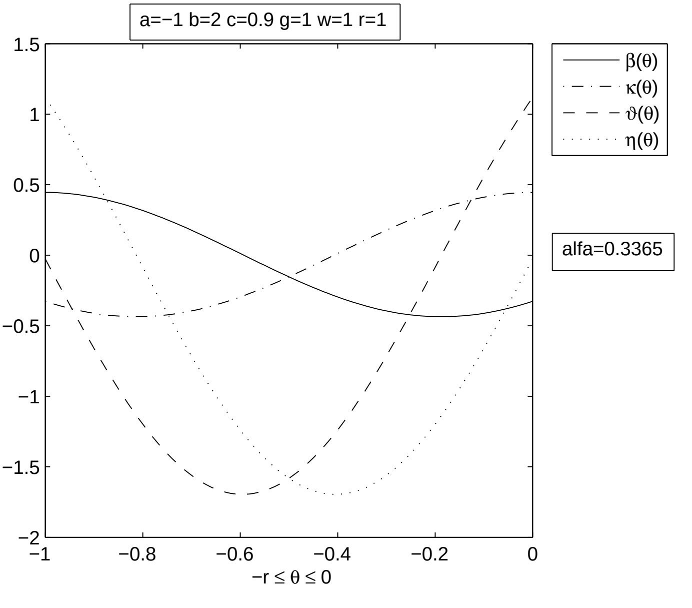

Fig. 3.1. Functions $\beta(\theta)$, $\kappa(\theta)$, $\vartheta(\theta)$, $\eta(\theta)$

3.4 A linear neutral system with a time-varying delay

3.4.1 Mathematical model of a linear neutral system with a time-varying delay

Let us consider a linear neutral system with a time-varying delay, whose dynamics is described by the functional-differential equation [18]

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-\tau(t))}{dt} = Ax(t) + Bx(t-\tau(t)) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.127}$$

where $t \geq t_0$, $\theta \in [-r, 0)$, $\tau(t)$ is a time-varying delay satisfying the condition $0 \leq \tau(t) \leq r$, $d\tau(t)/dt \neq 1$ where r is a positive constant A, B, $C \in \mathbb{R}^{n\times n}$ and C is non-singular, $x(t) \in \mathbb{R}^n$, $\varphi \in W^{1,2}([-r,0), \mathbb{R}^n)$. $W^{1,2}([-r,0), \mathbb{R}^n)$ is a space of all absolutely continuous functions $[-r,0) \to \mathbb{R}^n$ with derivatives in $L^2([-r,0), \mathbb{R}^n)$ a space of Lebesgue square integrable functions on an interval $[-r,0)$ with values in $\mathbb{R}^n$.
The space of initial data is given by the Cartesian product $\mathbb{R}^n \times W^{1,2}([-r,0), \mathbb{R}^n)$.
One can obtain a solution of FDE (3.127) using a step method. The step method is a basic method for solving FDE with a lumped delay. A solution is found on successive intervals, one after another, by solving an ordinary equation without delay in each interval.
The solution of equation (3.127) with initial value (x_0, φ) is an absolutely continuous function defined for $t \geq t_0$ with values in $\mathbb{R}^n$ and is denoted as $x(\cdot, t_0, (x_0, \varphi))$.

Definition 3.9. *The difference equation associated with (3.127) is given by*

$$x(t) = Cx(t-\tau(t)) \tag{3.128}$$

for $t \geq t_0$.

The eigenvalues of the difference equation (3.128) play a fundamental role in the asymptotic behavior of the solutions of neutral equation (3.127). The difference equation (3.128) is stable when the spectral radius $\gamma(C)$ of the matrix C fulfills the condition $\gamma(C) < 1$.
A new function y is introduced and defined by a term

$$y(t) = x(t) - Cx(t-\tau(t)) \tag{3.129}$$

for $t \geq t_0$.
Thus the equation (3.127) takes a form

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + (AC+B)x(t-\tau(t)) \\ y(t) = x(t) - Cx(t-\tau(t)) \\ y(t_0) = x_0 - C\varphi(-\tau(t)) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{3.130}$$

It is assumed that $\gamma(C) < 1$. Equation (3.130) can be written in the form

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + (AC+B)x_t(-\tau(t)) \\ y(t) = x(t) - Cx(t-\tau(t)) \\ y(t_0) = x_0 - C\varphi(-\tau(t)) \\ x_{t_0} = \Phi \end{cases} \tag{3.131}$$

where $x_t \in W^{1,2}([-r,0),\mathbb{R}^n)$ is a shifted restriction of the function $x(\cdot,t_0,(x_0,\varphi))$ to the interval $[-r,0)$. The state of system (3.131) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \tag{3.132}$$

for $t \geq t_0$, where $y(t) \in \mathbb{R}^n$, $x_t \in W^{1,2}([-r,0),\mathbb{R}^n)$.
The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r,0),\mathbb{R}^n) \tag{3.133}$$

The norm in the state space X is defined by the formula

$$\| S(t) \|_X = \sqrt{\| y(t) \|^2_{\mathbb{R}^n} + \| x_t \|^2_{W^{1,2}}} \tag{3.134}$$

for $t \geq t_0$.
In a parametric optimization problem is used an integral quadratic performance index of quality, which value is given by the term

$$J = \int_{t_0}^{\infty} y^T(t)Wy(t)dt = V(y_0,\varphi) \tag{3.135}$$

where V is the Lyapunov functional defined on the state space X and $W \in \mathbb{R}^{n\times n}$ is a positive definite matrix.

3.4.2 Determination of the Lyapunov functional

Let us consider a quadratic functional on $X \times [t_0,\infty)$, where X is defined by (3.133), given by the formula [18]

$$V(y(t),x_t,t) = y^T(t)\alpha(t)y(t) + \int_{-\tau(t)}^{0} y^T(t)\beta(\theta+\tau(t))x_t(\theta)d\theta +$$
$$+ \int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta+\tau(t),\sigma+\tau(t))x_t(\sigma)d\sigma d\theta \tag{3.136}$$

for $t \geq t_0$ where $\alpha \in C^1([t_0,\infty),\mathbb{R}^{n\times n})$, $\beta \in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$,
$\Omega = \{(\theta,\sigma) :\ \theta \in [0,\tau(t)],\ \sigma \in [\theta,0]\}$, $0 \leq \tau(t) \leq r$, where C^1 is a space of all continuous functions with continuous derivative.

In this paragraph will be given a procedure of determination of the functional (3.136) coefficients to obtain the Lyapunov functional.
The time derivative of the functional (3.136) on the trajectory of system (3.131) is computed. This time derivative is defined by the formula (2.10) which for system (3.131) takes a form

$$\frac{dV(y(t_0),\varphi,t_0)}{dt} = \limsup_{h\to 0}\frac{1}{h}\left[V\left(y(t_0+h),x_{t_0+h},t_0+h\right) - V\left(y(t_0),\varphi,t_0\right)\right] \tag{3.137}$$

It is taken the following procedure. One computes the time derivative of each term of the right-hand-side of the formula (3.136) and one substitutes in place of $dy(t)/dt$ and $\partial x_t(\theta)/\partial t$ the following terms

$$\frac{dy(t)}{dt} = Ay(t) + (AC+B)x_t(-\tau(t)) \tag{3.138}$$

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{3.139}$$

In such a manner one attains

$$\begin{aligned}
\frac{dV(y(t),x_t,t)}{dt} &= y^T(t)\left[A^T\alpha(t)+\alpha(t)A+\frac{d\alpha(t)}{dt}+\beta\left(\tau(t)\right)\right]y(t)+\\
&+y^T(t)\left[\left(\alpha(t)+\alpha^T(t)\right)(AC+B)+\beta(\tau(t))C+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)\right]x_t(-\tau(t))+\\
&+\int_{-\tau(t)}^{0} y^T(t)[A^T\beta(\theta+\tau(t))+\frac{d\beta(\theta+\tau(t))}{dt}-\frac{d\beta(\theta+\tau(t))}{d\theta}+\\
&+\delta^T(\theta+\tau(t),\tau(t))]x_t(\theta)d\theta+\int_{-\tau(t)}^{0} x_t^T(-\tau(t))\left[(AC+B)^T\beta(\theta+\tau(t))+\right.\\
&\left.+C^T\delta^T(\theta+\tau(t),\tau(t))+\delta(0,\theta+\tau(t))\left(\frac{d\tau(t)}{dt}-1\right)\right]x_t(\theta)d\theta+\\
&+\int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta)\left[\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta}+\right.\\
&\left.-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta
\end{aligned} \tag{3.140}$$

for $t \ge t_0$ where $\alpha \in C^1([t_0,\infty),\mathbb{R}^{n\times n})$, $\beta \in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$, $\Omega = \{(\theta,\sigma):\ \theta\in[0,\tau(t)],\ \sigma\in[\theta,0]\}$, $0 \le \tau(t) \le r$.
To achieve negative definiteness of that derivative we assume that

$$\frac{dV(y(t),x_t,t)}{dt} \equiv -y^T(t)Wy(t) \tag{3.141}$$

From relations (3.141) and (3.140) one attains the set of equations

$$A^T\alpha(t)+\alpha(t)A+\frac{d\alpha(t)}{dt}+\beta(\tau(t)) = -W \tag{3.142}$$

$$\left(\alpha(t)+\alpha^T(t)\right)(AC+B)+\beta\left(\tau(t)\right)C+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{3.143}$$

$$A^T\beta\left(\theta+\tau(t)\right)+\frac{d\beta\left(\theta+\tau(t)\right)}{dt}-\frac{d\beta\left(\theta+\tau(t)\right)}{d\theta}+\delta^T\left(\theta+\tau(t),\tau(t)\right)=0 \tag{3.144}$$

$$(AC+B)^T\beta\left(\theta+\tau(t)\right)+C^T\delta^T\left(\theta+\tau(t),\tau(t)\right)+\delta\left(0,\theta+\tau(t)\right)\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{3.145}$$

$$\frac{d\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{dt}-\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\theta}-\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\sigma}=0 \tag{3.146}$$

for $t\geq t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$ where $0\leq\tau(t)\leq r$.
The new variables are introduced

$$\xi=\theta+\tau(t) \tag{3.147}$$

$$\eta=\sigma+\tau(t) \tag{3.148}$$

The derivatives are calculated

$$\frac{d\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{dt}=\frac{d\delta(\xi,\eta)}{dt}=\frac{\partial\delta(\xi,\eta)}{\partial\xi}\frac{d\tau(t)}{dt}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}\frac{d\tau(t)}{dt} \tag{3.149}$$

$$\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\theta}=\frac{\partial\delta(\xi,\eta)}{\partial\theta}=\frac{\partial\delta(\xi,\eta)}{\partial\xi} \tag{3.150}$$

$$\frac{\partial\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)}{\partial\sigma}=\frac{\partial\delta(\xi,\eta)}{\partial\sigma}=\frac{\partial\delta(\xi,\eta)}{\partial\eta} \tag{3.151}$$

$$\frac{d\beta\left(\theta+\tau(t)\right)}{dt}=\frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial t}=\frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt} \tag{3.152}$$

$$\frac{d\beta\left(\theta+\tau(t)\right)}{d\theta}=\frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial\theta}=\frac{d\beta(\xi)}{d\xi} \tag{3.153}$$

The formula (3.146) takes a form

$$\frac{\partial\delta(\xi,\eta)}{\partial\xi}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}=0 \tag{3.154}$$

for $t\geq t_0$, $\theta\in[-\tau(t),0]$, $\sigma\in[\theta,0]$, $\xi\in[0,\tau(t)]$,$\eta\in[\xi,\tau(t)]$ where $0\leq\tau(t)\leq r$.
The formula (3.144) takes a form

$$\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta(\xi)}{d\xi}+A^T\beta(\xi)+\delta^T\left(\xi,\tau(t)\right)=0 \tag{3.155}$$

The formula (3.145) takes a form

$$(AC+B)^T \beta(\xi) + C^T \delta^T(\xi, \tau(t)) + \delta(0,\xi)\left(\frac{d\tau(t)}{dt} - 1\right) = 0 \tag{3.156}$$

The solution of equation (3.146) is given by the formula

$$\delta(\theta + \tau(t), \sigma + \tau(t)) = \delta(\xi, \eta) = f(\xi - \eta) = f(\theta - \sigma) \tag{3.157}$$

for $t \geq t_0$, $\theta \in [-\tau(t), 0]$, $\sigma \in [\theta, 0]$, $0 \leq \tau(t) \leq r$ where $f \in C^1([-r,r], \mathbb{R}^{n\times n})$
The formula (3.155) implies

$$\delta^T(\xi, \tau(t)) = f^T(\xi - \tau(t)) = -\left(\frac{d\tau(t)}{dt} - 1\right)\frac{d\beta(\xi)}{d\xi} - A^T \beta(\xi) \tag{3.158}$$

One puts the term (3.158) into (3.156). After calculations one obtains

$$C^T \frac{d\beta(\xi)}{d\xi} = \left(\frac{d\tau(t)}{dt} - 1\right)^{-1} B^T \beta(\xi) + \delta(0,\xi) \tag{3.159}$$

From the relation (3.158) one can determine the term $\delta(0,\xi) = f(-\xi)$

$$f(-\xi) = \left(\frac{d\tau(t)}{dt} - 1\right)\frac{d\beta^T(-\xi + \tau(t))}{d\xi} - \beta^T(-\xi + \tau(t))A \tag{3.160}$$

and put it into (3.159). In this way the relation is obtained

$$C^T \frac{d\beta(\xi)}{d\xi} - \left(\frac{d\tau(t)}{dt} - 1\right)\frac{d\beta^T(-\xi + \tau(t))}{d\xi} =$$

$$= \left(\frac{d\tau(t)}{dt} - 1\right)^{-1} B^T \beta(\xi) - \beta^T(-\xi + \tau(t))A \tag{3.161}$$

for $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$
Into the formula (3.161) instead of ξ one substitutes the new variable $-\xi + \tau(t)$. After calculations the formula is attained

$$\left(\frac{d\tau(t)}{dt} - 1\right)\frac{d\beta(\xi)}{d\xi} - \frac{d\beta^T(-\xi + \tau(t))}{d\xi} C =$$

$$= \left(\frac{d\tau(t)}{dt} - 1\right)^{-1} \beta^T(-\xi + \tau(t))B - A^T \beta(\xi) \tag{3.162}$$

In this way one obtained the set of differential equations

$$\begin{cases} C^T \dfrac{d\beta(\xi)}{d\xi} - \left(\dfrac{d\tau(t)}{dt} - 1\right) \dfrac{d\beta^T(-\xi+\tau(t))}{d\xi} = \\ = \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} B^T\beta(\xi) - \beta^T(-\xi+\tau(t))A \\ \\ \left(\dfrac{d\tau(t)}{dt} - 1\right) \dfrac{d\beta(\xi)}{d\xi} - \dfrac{d\beta^T(-\xi+\tau(t))}{d\xi}C = \\ = \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} \beta^T(-\xi+\tau(t))B - A^T\beta(\xi) \end{cases} \tag{3.163}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$ with the initial conditions $\beta(0)$ and $\beta(\tau(t))$. One can reshape the set of equations (3.163) to the form

$$\begin{cases} C^T \dfrac{d\beta(\xi)}{d\xi}C - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2 \dfrac{d\beta(\xi)}{d\xi} = \left(\dfrac{d\tau(t)}{dt} - 1\right)A^T\beta(\xi) + \\ + \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} B^T\beta(\xi)C - \beta^T(-\xi+\tau(t))(AC+B) \\ \\ C^T \dfrac{d\beta(-\xi+\tau(t))}{d\xi}C - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2 \dfrac{d\beta(-\xi+\tau(t))}{d\xi} = \beta^T(\xi)(AC+B) + \\ - \left(\dfrac{d\tau(t)}{dt} - 1\right)A^T\beta(-\xi+\tau(t)) - \left(\dfrac{d\tau(t)}{dt} - 1\right)^{-1} B^T\beta(-\xi+\tau(t))C \end{cases} \tag{3.164}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$ with the initial conditions $\beta(0)$ and $\beta(\tau(t))$ There holds the relationship between $\beta(\xi)$ and $\beta(-\xi+\tau(t))$

$$\beta(\xi)\,|_{\xi=\frac{\tau(t)}{2}} = \beta(-\xi+\tau(t))\,|_{\xi=\frac{\tau(t)}{2}} \tag{3.165}$$

The derivative of the equation (3.143) with respect to t is calculated

$$\left(\frac{d\alpha(t)}{dt} + \frac{d\alpha^T(t)}{dt}\right)(AC+B) + \frac{d\beta(\tau(t))}{dt}C + \frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right) + \frac{d^2\tau(t)}{dt^2}\beta(0) = 0 \tag{3.166}$$

where

$$\frac{d\beta(0)}{dt} = \frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt}\,|_{\xi=0} \tag{3.167}$$

$$\frac{d\beta(\tau(t))}{dt} = \frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt}\,|_{\xi=\tau(t)} \tag{3.168}$$

Equation (3.164) implies

$$\begin{aligned} C^T \frac{d\beta(0)}{dt}C - \left(\frac{d\tau(t)}{dt} - 1\right)^2 \frac{d\beta(0)}{dt} = \frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)A^T\beta(0) + \\ + \frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)^{-1} B^T\beta(0)C - \frac{d\tau(t)}{dt}\beta^T(\tau(t))(AC+B) \end{aligned} \tag{3.169}$$

$$C^T\frac{d\beta(\tau(t))}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(\tau(t))}{dt}=\frac{d\tau(t)}{dt}\beta^T(0)(AC+B)+$$
$$-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(\tau(t))-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))C \tag{3.170}$$

From equation (3.142) one obtains

$$\frac{d\alpha(t)}{dt}=-A^T\alpha(t)-\alpha(t)A-\beta(\tau(t))-G \tag{3.171}$$

One puts the term (3.171) into the equation (3.166). After calculations one gets

$$\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right](AC+B)+\left(\beta(\tau(t))+\beta^T(\tau(t))\right)(AC+B)+$$
$$-\frac{d^2\tau(t)}{dt^2}\beta(0)-\frac{d\beta(\tau(t))}{dt}C-\frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)=-\left(G+G^T\right)(AC+B) \tag{3.172}$$

The matrix $\alpha(t)$, the initial conditions of system (3.164) and $d\beta(0)/dt$, $d\beta(\tau(t))/dt$ are obtained by solving the set of algebraic equations (3.172), (3.143), (3.169), (3.176) and (3.165). That set of equations is written below

$$\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right](AC+B)+\left(\beta(\tau(t))+\beta^T(\tau(t))\right)(AC+B)+$$
$$-\frac{d^2\tau(t)}{dt^2}\beta(0)-\frac{d\beta(\tau(t))}{dt}C-\frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)=-\left(G+G^T\right)(AC+B) \tag{3.173}$$

$$\left(\alpha(t)+\alpha^T(t)\right)(AC+B)+\beta\left(\tau(t)\right)C+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)=0 \tag{3.174}$$

$$C^T\frac{d\beta(0)}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(0)}{dt}=\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(0)+$$
$$+\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(0)C-\frac{d\tau(t)}{dt}\beta^T\left(\tau(t)\right)(AC+B) \tag{3.175}$$

$$C^T\frac{d\beta(\tau(t))}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(\tau(t))}{dt}=\frac{d\tau(t)}{dt}\beta^T(0)(AC+B)+$$
$$-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(\tau(t))-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))C \tag{3.176}$$

$$\beta(\xi)\,|_{\xi=\frac{\tau(t)}{2}}=\beta(-\xi+\tau(t))\,|_{\xi=\frac{\tau(t)}{2}} \tag{3.177}$$

Having the solution of the set of differential equations (3.164) and taking into account the formulas (3.147), (3.157) and (3.160) one can get the matrices

$$\beta(\theta+\tau(t)) = \beta(\xi)\,|_{\xi=\theta+\tau(t)} \tag{3.178}$$

$$\delta(\theta+\tau(t),\sigma+\tau(t)) = f(\sigma-\theta) \tag{3.179}$$

where

$$f(\rho) = -\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta^T(\rho+\tau(t))}{d\rho} - \beta^T(\rho+\tau(t))A \tag{3.180}$$

for $t \geq t_0$; $\theta \in [-\tau(t),0]$; $\sigma \in [\theta,0]$ where $0 \leq \tau(t) \leq r$.
In this way one obtained all coefficients of the functional (3.136). This coefficients depend on the matrices A, B and C of system (3.131). The time derivative of the functional (3.136) is negative definite.

3.4.3 The example. Inertial system with delay and a PD controller

Let us consider a first order inertial system with delay described by the equation [19]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{q}{T}x(t) + \dfrac{k_0}{T}u(t-\tau(t)) \\ x(t_0) = x_o \\ x(t_0+\theta) = 0 \\ u(t) = -px(t) - T_d\dfrac{dx(t)}{dt} \end{cases} \tag{3.181}$$

$t \geq t_0$, $x(t) \in \mathbb{R}$, $\theta \in [-r,0)$, p, k_0, T, T_d, q, $x_0 \in \mathbb{R}$, $\tau(t)$ is a time-varying delay satisfying the condition $0 \leq \tau(t) \leq r$, $d\tau(t)/dt \neq 1$ where r is positive constant. The parameter k_0 is a gain of a plant, p is a proportional gain, T_d is a derivative gain, T is a system time constant, x_0 is an initial state of system. In the case $q=1$ the equation (3.181) describes a static object and in the case $q=0$ the equation (3.181) describes an astatic object.
One can reshape equation (3.181) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} + \dfrac{k_0T_d}{T}\dfrac{dx(t-\tau(t))}{dt} = -\dfrac{q}{T}x(t) - \dfrac{k_0p}{T}x(t-\tau(t)) \\ x(t_0) = x_o \\ x(\theta) = 0 \end{cases} \tag{3.182}$$

for $t \geq t_0$ and $\theta \in [-r,0)$.
It is assumed that the element k_0T_d/T satisfies the condition (3.14), whose takes a form

$$\left|\frac{k_0T_d}{T}\right| < 1 \tag{3.183}$$

A new function y is introduced and defined by the term

$$y(t) = x(t) - Cx(t - \tau(t)) \tag{3.184}$$

for $t \geq t_0$.
One can reshape equation (3.182) to the form

$$\begin{cases} \dfrac{dy(t)}{dt} = -\dfrac{q}{T}y(t) + \left(\dfrac{qk_0T_d}{T^2} - \dfrac{k_0p}{T}\right)x(t-\tau(t)) \\ y(t) = x(t) + \dfrac{k_0T_d}{T}x(t-\tau(t)) \\ y(t_0) = x_0 \\ x(t_0+\theta) = 0 \end{cases} \tag{3.185}$$

Performance index of quality has a form

$$J = \int_{t_0}^{\infty} y^2(t)dt = V(y(t_0), \varphi, t_0) \tag{3.186}$$

The Lyapunov functional is given by the formula

$$V(y(t), x_t, t) = \alpha(t)y^2(t) + \int_{-\tau(t)}^{0} \beta(\theta+\tau(t))\, y(t)\, x_t(\theta)\, d\theta +$$

$$+ \int_{-\tau(t)}^{0} \int_{\theta}^{0} \delta(\theta+\tau(t), \sigma+\tau(t))\, x_t(\theta)\, x_t(\sigma)\, d\sigma d\theta \tag{3.187}$$

where

$$x_t(\theta) = x(t+\theta)$$

for $\theta \in [-r, 0)$, $x_t \in W^{1,2}([-r,0),\mathbb{R})$
The coefficients of the functional (3.187) will be obtained.
Equation (3.164) takes the form

$$\begin{bmatrix} \dfrac{d\beta(\xi)}{d\xi} \\ \dfrac{d\beta(-\xi+\tau(t))}{d\xi} \end{bmatrix} = \begin{bmatrix} p_1 & -p_2 \\ p_2 & -p_1 \end{bmatrix} \begin{bmatrix} \beta(\xi) \\ \beta(-\xi+\tau(t)) \end{bmatrix} \tag{3.188}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$, where $0 \leq \tau(t) \leq r$

$$p_1 = \frac{-\dfrac{q}{T}\left(\dfrac{d\tau(t)}{dt} - 1\right) + \dfrac{k_0^2pT_d}{T^2\left(\dfrac{d\tau(t)}{dt} - 1\right)}}{\dfrac{k_0^2T_d^2}{T^2} - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2} \tag{3.189}$$

$$p_2 = \frac{\dfrac{qk_0T_d}{T^2} - \dfrac{k_0 p}{T}}{\dfrac{k_0^2T_d^2}{T^2} - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2} \tag{3.190}$$

The fundamental matrix of the differential equation (3.188) is given by the formula

$$R(\xi) = \begin{bmatrix} ch\lambda\xi + \frac{p_1}{\lambda} sh\lambda\xi & -\dfrac{p_2}{\lambda} sh\lambda\xi \\ \dfrac{p_2}{\lambda} sh\lambda\xi & ch\lambda\xi - \dfrac{p_1}{\lambda} sh\lambda\xi \end{bmatrix} \tag{3.191}$$

where

$$\lambda = \frac{\sqrt{\dfrac{k_0^2p^2 - q^2\left(\frac{d\tau(t)}{dt} - 1\right)^2}{\dfrac{k_0^2T_d^2}{T^2} - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2}}}{T\left(\dfrac{d\tau(t)}{dt} - 1\right)} \tag{3.192}$$

Hence

$$\begin{bmatrix} \beta(\xi) \\ \beta(-\xi + \tau(t)) \end{bmatrix} = R(\xi) \begin{bmatrix} \beta(0) \\ \beta(\tau(t)) \end{bmatrix} \tag{3.193}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$.
One needs the initial conditions of the set of differential equations (3.188) to obtain

$$\beta(\theta + \tau(t)) = \beta(\xi)\,|_{\xi = \theta + \tau(t)} \tag{3.194}$$

$$\delta(\theta + \tau(t), \sigma + \tau(t)) = f(\sigma - \theta) \tag{3.195}$$

$$f(\rho) = -\left(\frac{d\tau(t)}{dt} - 1\right)\frac{d\beta(\rho + \tau(t))}{d\rho} - a\beta(\rho + \tau(t)) \tag{3.196}$$

for $t \geq t_0$, $\theta \in [-\tau(t), 0]$, $\sigma \in [\theta, 0]$ where $0 \leq \tau(t) \leq r$.
The initial conditions of the differential equation (3.188) and the coefficient $\alpha(t)$ are attained by solving of the set of equations (3.173) to (3.177) which take the form as below

$$\begin{aligned} &4a\left(\frac{qk_0T_d}{T^2} - \frac{k_0p}{T}\right)\alpha(t) + \left(\frac{k_0T_d}{T}p_2\frac{d\tau(t)}{dt} - \frac{d^2\tau(t)}{dt^2} - p_1\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)\right)\beta(0) + \\ &+ \left(2\left(\frac{qk_0T_d}{T^2} - \frac{k_0p}{T}\right) - \frac{k_0T_d}{T}p_1\frac{d\tau(t)}{dt} + p_2\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)\right)\beta(\tau(t)) = \\ &= -2w\left(\frac{qk_0T_d}{T^2} - \frac{k_0p}{T}\right) \end{aligned} \tag{3.197}$$

$$2\left(\frac{qk_0T_d}{T^2}-\frac{k_0p}{T}\right)\alpha(t)+\left(\frac{d\tau(t)}{dt}-1\right)\beta(0)-\frac{k_0T_d}{T}\beta(\tau(t))=0 \tag{3.198}$$

$$\left(ch\frac{\lambda\tau(t)}{2}+\frac{p_1-p_2}{\lambda}sh\frac{\lambda\tau(t)}{2}\right)\beta(0)+ \\ +\left(\frac{p_1-p_2}{\lambda}sh\frac{\lambda\tau(t)}{2}-ch\frac{\lambda\tau(t)}{2}\right)\beta(\tau(t))=0 \tag{3.199}$$

We compute the value of the performance index (3.186) for initial conditions given below

$$\begin{cases} y(0)=x_0 \\ \varphi(\theta)=0 \end{cases}$$

for $\theta\in[-r,0)$

$$J(t)=x_0^2\alpha(t)$$

for $t\geq 0$.
Figures show the graphs of functions $J(t)$, $\beta(\xi)$ and $\beta(-\xi+\tau(t))$ obtained with the Matlab code, for given values of parameters $q=1$, $T=5$, $k_0=1$, $x_0=1$ and $\tau(t)=r(1-exp(-\frac{t}{3}))$, $r=0.5$ of system (3.185). Figure 3.2 presents the index value graph for $p=6.9003$ and $T_d=-4.6802$. These values are called the critical values of p and T_d. For p and T_d greater then critical ones system (3.185) becomes unstable.

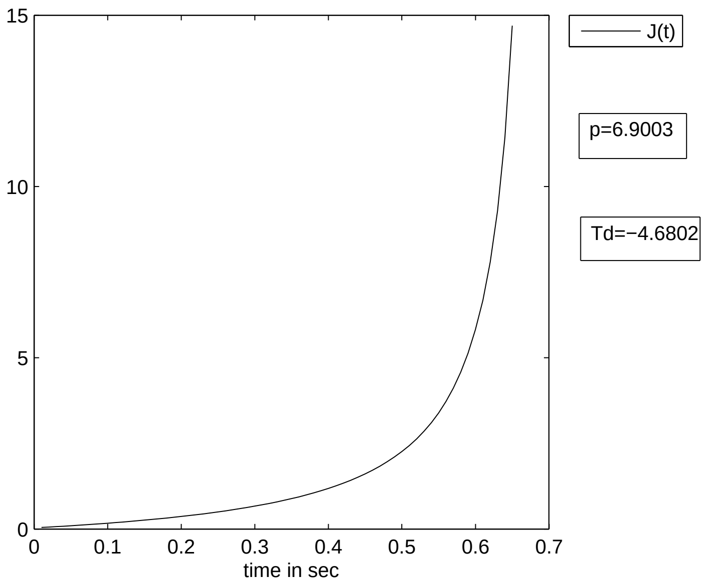

Fig. 3.2. Value of the index $J(t)$ for $p=6.9003$ and $T_d=-4.6802$

Figures 3.3–3.5 show the functions $J(t)$, $\beta(\xi)$ and $\beta(-\xi+\tau(t))$ for $p=5$ and $T_d=-2$.

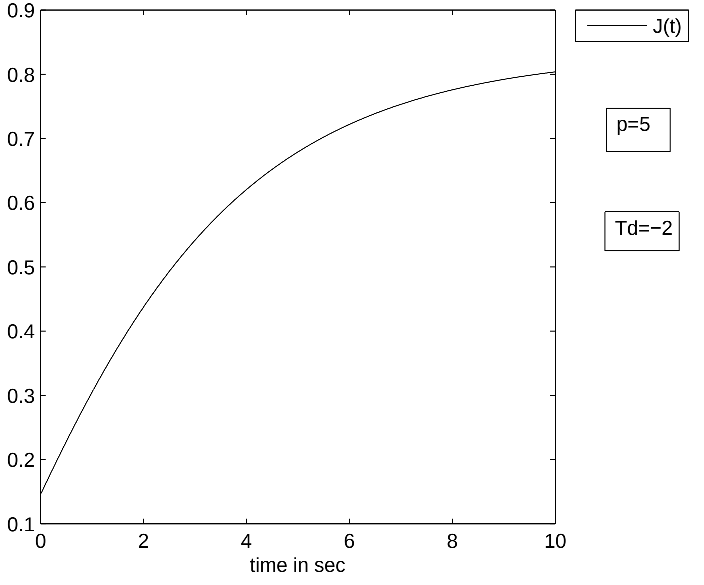

Fig. 3.3. Value of the index $J(t)$ for $p=5$ and $T_d=-2$

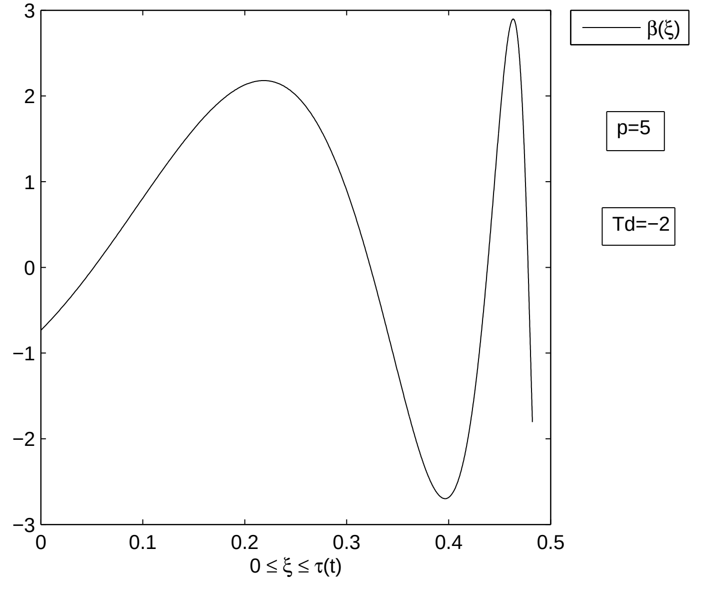

Fig. 3.4. Function $\beta(\xi)$

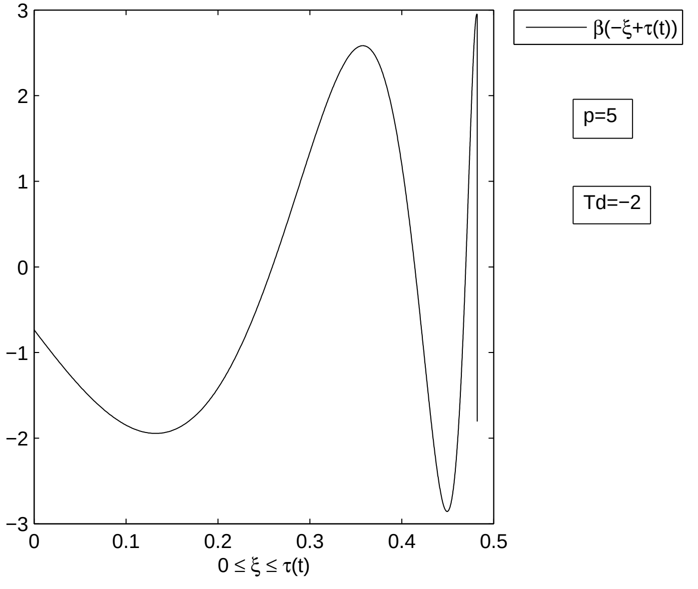

Fig. 3.5. Function $\beta(-\xi + \tau(t))$

4 The Lyapunov matrix for a retarded type time delay system

4.1 Mathematical model of a retarded type time delay system

Let us consider a time-delay system

$$\begin{cases} \frac{dx(t)}{dt} = \sum_{j=0}^{m} A_j x(t-h_j) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{4.1}$$

for $t \geq t_0,\ \theta \in [-h, 0]$

Where $x(t) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n\times n}$, $0 = h_0 < h_1 < ... < h_m = h$, function $\varphi \in PC([-h,0],\mathbb{R}^n)$ – the space of piece-wise continuous vector valued functions defined on the segment $[-h,0]$ with the uniform norm $\| \varphi \|_{PC} = \sup\limits_{\theta\in[-h,0]} \| \varphi(\theta) \|$

Let $x(t,t_0,\varphi)$ be the solution of system (4.1) with the initial function φ.

Definition 4.1. [2] *$K(t)$ is the* ***fundamental matrix*** *of system (4.1) if it satisfies the matrix equation*

$$\frac{d}{dt}K(t) = \sum_{j=0}^{m} A_j K(t-h_j)$$

for $t \geq 0$ and the following initial condition $K(0) = I_{n\times n}$ and $K(t) = 0_{n\times n}$ for $t < 0$ where $I_{n\times n}$ is the identity $n\times n$ matrix and $0_{n\times n}$ is the zero $n\times n$ matrix.

Theorem 4.1. [2] *Let $K(t)$ be the fundamental matrix of system (4.1), then for $t \geq t_0$*

$$x(t,t_0,\varphi) = K(t-t_0)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t-t_0-h_j-\theta)A_j\varphi(\theta)d\theta \tag{4.2}$$

The initial condition holds

$$x_{t_0}(t_0,\varphi)=\varphi \tag{4.3}$$

for $\theta \in [-h,0]$, where $x_t(t_0,\varphi) \in PC([-h,0],\mathbb{R}^n)$ is a shifted restriction of the function $x(\cdot,t_0,\varphi)$ to the segment $[-h,0]$.

4.2 The Lyapunov–Krasovskii functional for a retarded type time delay system

Given a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$. We are looking for a functional

$$v : PC([-h,0],\mathbb{R}^n) \to \mathbb{R}$$

such that along the solutions of system (4.1) the following equality holds

$$\frac{d}{dt}v(x_t(t_0,\varphi)) = -x^T(t,t_0,\varphi)Wx(t,t_0,\varphi) \tag{4.4}$$

for $t \geq t_0$, where $x(t,t_0,\varphi)$ is a solution of system (4.1), with the initial function $\varphi \in PC([-h,0],\mathbb{R}^n)$, given by (4.2).
We assume that system (4.1) is asymptotically stable and integrate both side of equation (4.4) from t_0 to infinity. We obtain

$$v(x_{t_0}(t_0,\varphi)) = v(\varphi) = \int_{t_0}^{\infty} x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt \tag{4.5}$$

Taking into account (4.2) we calculate the integral of the right-hand side of equation (4.5)

$$\int_{t_0}^{\infty} x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt = \varphi^T(0)\int_0^{\infty} K^T(t)WK(t)dt\varphi(0)+$$
$$+\sum_{j=1}^{m}\int_{-h_j}^{0} 2\varphi^T(0)\int_0^{\infty} K^T(t)WK(t-h_j-\theta)dtA_j\varphi(\theta)d\theta+$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{m}\int_{-h_j}^{0}\varphi^T(\theta)A_j^T\int_{-h_k}^{0}\int_0^{\infty}K^T(t-h_j-\theta)WK(t-h_k-\eta)dtA_k\varphi(\eta)d\eta d\theta \tag{4.6}$$

The relations hold

$$\int_0^\infty K^T(t-h_j-\theta)WK(t-h_k-\eta)dt = \int_{-h_j-\theta}^\infty K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma =$$

$$= \int_{-h_j-\theta}^0 K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma + \int_0^\infty K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma =$$

$$= \int_0^\infty K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma$$

The term

$$\int_{-h_j-\theta}^0 K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma = 0$$

because $K(\varsigma) = 0$ for $\varsigma < 0$. Formula (4.6) takes a form

$$\int_{t_0}^\infty x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt = \varphi^T(0)\int_0^\infty K^T(t)WK(t)dt\varphi(0)+$$

$$+\sum_{j=1}^m \int_{-h_j}^0 2\varphi^T(0)\int_0^\infty K^T(t)WK(t-h_j-\theta)dtA_j\varphi(\theta)d\theta+$$

$$+\sum_{j=1}^m\sum_{k=1}^m \int_{-h_j}^0 \varphi^T(\theta)A_j^T \int_{-h_k}^0 \int_0^\infty K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma A_k\varphi(\eta)d\eta d\theta \qquad (4.7)$$

Definition 4.2. [81] *We introduce* ***a Lyapunov matrix***

$$U(\xi) = \int_0^\infty K^T(t)WK(t+\xi)dt \qquad (4.8)$$

for $\xi \geq 0$.

Using the Lyapunov matrix (4.8) and taking into account equation (4.5) we obtain the formula for the functional $v(\varphi)$

$$v(\varphi) = \int_{t_0}^\infty x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt = \varphi^T(0)U(0)\varphi(0)+$$

$$+2\varphi^T(0)\sum_{j=1}^m \int_{-h_j}^0 U(-\theta-h_j)A_j\varphi(\theta)d\theta+$$

$$+\sum_{j=1}^m\sum_{k=1}^m \int_{-h_j}^0 \int_{-h_k}^0 \varphi^T(\theta)A_j^T U(h_j-h_k+\theta-\eta)A_k\varphi(\eta)d\eta d\theta \qquad (4.9)$$

Corollary 4.1. *The Lyapunov matrix (4.8) satisfies the following properties* [81]*:*
Dynamic property

$$\frac{d}{d\xi}U(\xi) = \sum_{j=0}^{m} U(\xi - h_j)A_j \tag{4.10}$$

for $\xi \geq 0$
Symmetry property

$$U(-\xi) = U^T(\xi) \tag{4.11}$$

for $\xi \geq 0$
Algebraic property

$$\sum_{j=0}^{m} [U(-h_j)A_j + A_j^T U(h_j)] = -W \tag{4.12}$$

Formulas (4.10), (4.11), (4.12) enable us to calculate the Lyapunov matrix $U(\xi)$ for $\xi \geq 0$.

4.3 The Lyapunov matrix for a system with one delay

Let us consider a system [20]

$$\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.13}$$

for $t \geq 0$ and $\theta \in [-h, 0]$. Where $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\varphi \in PC([-h,0],\mathbb{R}^n)$, $0 < h \in \mathbb{R}$.
System of equations (4.10), (4.11), (4.12) takes a form

$$\frac{d}{d\xi}U(\xi) = U(\xi)A_0 + U(\xi - h)A_1 \tag{4.14}$$

$$U(-\xi) = U^T(\xi) \tag{4.15}$$

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W \tag{4.16}$$

for $\xi \in [0, h]$.
Formula (4.15) implies

$$U(\xi - h) = U^T(h - \xi) = Z(\xi) \tag{4.17}$$

We compute the derivative of $Z(\xi)$

$$\frac{d}{d\xi}Z(\xi) = \frac{d}{d\xi}U^T(h-\xi) = -A_0^T U^T(h-\xi) - A_1^T U^T(-\xi) = -A_0^T Z(\xi) - A_1^T U(\xi) \tag{4.18}$$

We have received the set of ordinary differential equations

$$\begin{cases} \dfrac{d}{d\xi}U(\xi) = U(\xi)A_0 + Z(\xi)A_1 \\ \\ \dfrac{d}{d\xi}Z(\xi) = -A_0^T Z(\xi) - A_1^T U(\xi) \end{cases} \tag{4.19}$$

for $\xi \in [0,h]$ with initial condition $U(0)$, $Z(0)$.
Formula (4.17) implies

$$U(-h) = U^T(h) = Z(0) \tag{4.20}$$

Taking (4.20) into account equation (4.16) takes a form

$$U(0)A_0 + Z(0)A_1 + A_0^T U(0) + A_1^T Z^T(0) = -W \tag{4.21}$$

Using the Kronecker product we can express (4.19) in a form

$$\begin{bmatrix} \dfrac{d}{d\xi} colU(\xi) \\ \\ \dfrac{d}{d\xi} colZ(\xi) \end{bmatrix} = \begin{bmatrix} A_0^T \otimes I & A_1^T \otimes I \\ \\ -I \otimes A_1^T & -I \otimes A_0^T \end{bmatrix} \begin{bmatrix} colU(\xi) \\ \\ colZ(\xi) \end{bmatrix} \tag{4.22}$$

for $\xi \in [0,h]$ with initial condition $colU(0)$, $colZ(0)$.
Formula (4.21) can be expressed

$$(A_0^T \otimes I + I \otimes A_0^T)colU(0) + (A_1^T \otimes I)colZ(0) + (I \otimes A_1^T)colZ^T(0) = -colW \tag{4.23}$$

Solution of equation (4.32) is given by a term

$$\begin{bmatrix} colU(\xi) \\ \\ colZ(\xi) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix} \begin{bmatrix} colU(0) \\ \\ colZ(0) \end{bmatrix} \tag{4.24}$$

where a matrix $\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix}$ is a fundamental matrix of system (4.22).

We determine the initial conditions $colU(0)$, $colZ(0)$.
The term (4.17) implies $Z(h) = U^T(0) = U(0)$.
From (4.24) we obtain

$$colU(h) = colZ^T(0) = \Phi_{11}(h)colU(0) + \Phi_{12}(h)colZ(0) \tag{4.25}$$

$$colZ(h) = colU(0) = \Phi_{21}(h)colU(0) + \Phi_{22}(h)colZ(0) \tag{4.26}$$

We put (4.25) into (4.23) and reshape (4.26). In this way we attain the set of algebraic equations which enables us to calculate the initial conditions of (4.24).

$$\begin{aligned} &\left[A_0^T \otimes I + I \otimes A_0^T + (I \otimes A_1^T)\Phi_{11}(h)\right] colU(0) + \\ &+ \left[A_1^T \otimes I + (I \otimes A_1^T)\Phi_{12}(h)\right] colZ(0) = -colW \end{aligned} \tag{4.27}$$

$$\left[I - \Phi_{21}(h)\right] colU(0) - \Phi_{22}(h) colZ(0) = 0 \tag{4.28}$$

4.4 Formulation of the parametric optimization problem for a system with one delay

Let us consider a time-delay system with a P-controller

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bu(t-h) \\ u(t) = -Px(t) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.29}$$

for $t \geq 0$ and $\theta \in [-h, 0]$. Where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $P \in \mathbb{R}^{p\times n}$ is a P-controller gain and $\varphi \in PC([-h,0],\mathbb{R}^n)$, $0 < h \in \mathbb{R}$.
System (4.29) can be written in the equivalent form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) - BPx(t-h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.30}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_0^\infty x^T(t;\varphi) W x(t;\varphi) dt \tag{4.31}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t;\varphi)$ is a solution of (4.30) for initial function φ.

Problem 4.1. *Determine the matrix $P \in \mathbb{R}^{p\times n}$ whose minimize an integral quadratic performance index of quality (4.31)*

According to (4.4) the value of the performance index of quality (4.31) is equal to the value of the functional (4.9) for initial function φ. To calculate the value of the functional (4.9) we need a Lyapunov matrix $U(\xi)$.

To obtain a Lyapunov matrix $U(\xi)$ we solve a system of differential equations (4.22) and a set of algebraic equations (4.27) and (4.28) whose take a form

$$\begin{bmatrix} \frac{d}{d\xi} colU(\xi) \\ \frac{d}{d\xi} colZ(\xi) \end{bmatrix} = \begin{bmatrix} A^T \otimes I & -P^T B^T \otimes I \\ I \otimes P^T B^T & -I \otimes A^T \end{bmatrix} \begin{bmatrix} colU(\xi) \\ colZ(\xi) \end{bmatrix} \tag{4.32}$$

$$\left[A^T \otimes I + I \otimes A^T - (I \otimes P^T B^T)\Psi_{11}(h)\right] colU(0) +$$

$$-\left[P^T B^T \otimes I + (I \otimes P^T B^T)\Psi_{12}(h)\right] colZ(0) = -colW \tag{4.33}$$

$$\left[I - \Psi_{21}(h)\right] colU(0) - \Psi_{22}(h) colZ(0) = 0 \tag{4.34}$$

where $\Psi(\xi) = \begin{bmatrix} \Psi_{11}(\xi) & \Psi_{12}(\xi) \\ \Psi_{21}(\xi) & \Psi_{22}(\xi) \end{bmatrix}$ is the fundamental matrix of system (4.32).

4.5 The examples

4.5.1 Inertial system with delay and a P-controller

Let us consider inertial system with delay and a P-controller [20]

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{T}x(t) + \frac{k_0}{T}u(t-h) \\ u(t) = -px(t) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{4.35}$$

$t \geq 0$, $x(t) \in \mathbb{R}$, $\theta \in [-h, 0)$, p, k_0, T, $x_0 \in \mathbb{R}$, $h \geq 0$. The parameter k_0 is a gain of a plant, p is a gain of a P-controller, T is a system time constant, x_0 is an initial state of system.
One can reshape equation (4.35) to a form

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{T}x(t) - \frac{k_0 p}{T}x(t-r) \\ x(0) = x_o \\ x(\theta) = 0 \end{cases} \tag{4.36}$$

for $t \geq 0$ and $\theta \in [-h, 0)$.
The initial function φ has a form

$$\varphi(\theta) = \begin{cases} x_0 & \text{for} \quad \theta = 0 \\ 0 & \text{for} \quad \theta \in [-h, 0) \end{cases} \tag{4.37}$$

In parametric optimization problem we use the performance index

$$J=\int_0^\infty wx^2(t;\varphi)dt \tag{4.38}$$

where $w>0$ and $x(t;\varphi)$ is a solution of (4.36) for initial function (4.37).
The differential equation (4.32) takes a form

$$\begin{bmatrix} \dfrac{d}{d\xi}U(\xi) \\ \dfrac{d}{d\xi}Z(\xi) \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{T} & -\dfrac{k_0 p}{T} \\ \dfrac{k_0 p}{T} & \dfrac{1}{T} \end{bmatrix} \begin{bmatrix} U(\xi) \\ Z(\xi) \end{bmatrix} \tag{4.39}$$

The fundamental matrix of (4.39) is given

$$\Phi(\xi)=\begin{bmatrix} \cosh\lambda\xi-\dfrac{1}{\lambda T}\sinh\lambda\xi & -\dfrac{k_0 p}{\lambda T}\sinh\lambda\xi \\ \dfrac{k_0 p}{\lambda T}\sinh\lambda\xi & \cosh\lambda\xi+\dfrac{1}{\lambda T}\sinh\lambda\xi \end{bmatrix} \tag{4.40}$$

for $\xi\in[0,h]$, where

$$\lambda=\frac{1}{T}\sqrt{1-k_0^2p^2} \tag{4.41}$$

The initial conditions for (4.39) are obtained from equations (4.33) and (4.34) which take a form

$$\begin{bmatrix} 2+k_0p(\cosh\lambda h-\dfrac{1}{\lambda T}\sinh\lambda h) & k_0p(1-\dfrac{k_0p}{\lambda T}\sinh\lambda h) \\ 1-\dfrac{k_0p}{\lambda T}\sinh\lambda h & -\cosh\lambda h-\dfrac{1}{\lambda T}\sinh\lambda h \end{bmatrix} \begin{bmatrix} U(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} Tw \\ 0 \end{bmatrix} \tag{4.42}$$

Solving (4.42) we obtain

$$U(0)=\frac{\dfrac{Tw}{2}\left(\cosh\lambda h+\dfrac{1}{\lambda T}\sinh\lambda h\right)}{k_0p+\cosh\lambda h+\lambda T\sinh\lambda h} \tag{4.43}$$

$$Z(0)=\frac{\dfrac{Tw}{2}\left(1-\dfrac{k_0p}{\lambda T}\sinh\lambda h\right)}{k_0p+\cosh\lambda h+\lambda T\sinh\lambda h} \tag{4.44}$$

The solution of (4.39) is given

$$U(\xi)=\frac{w}{2}\left(\frac{T\cosh\lambda h+\dfrac{1}{\lambda}\sinh\lambda h}{k_0p+\cosh\lambda h+\lambda T\sinh\lambda h}\cosh\lambda\xi-\frac{1}{\lambda}\sinh\lambda\xi\right) \tag{4.45}$$

$$Z(\xi) = \frac{\frac{Tw}{2}}{k_0 p + \cosh\lambda h + \lambda T \sinh\lambda h}\Bigg(\Big(1 - \frac{k_0 p}{\lambda T}\sinh\lambda h\Big)\cosh\lambda\xi + \\ + \frac{1}{\lambda T}\Big(1 + k_0 p \cosh\lambda h\Big)\sinh\lambda\xi\Bigg) \tag{4.46}$$

The value of the performance index (4.38) is equal to the value of functional (4.9) for $U(\xi)$ given by (4.45) and initial function given by (4.37)

$$J = \frac{\frac{Tw}{2}\Big(\cosh\lambda h + \frac{1}{\lambda T}\sinh\lambda h\Big)}{k_0 p + \cosh\lambda h + \lambda T \sinh\lambda h} x_0^2 \tag{4.47}$$

Figure 4.1 shows the value of the index $J(p)$ for $x_0 = 1$, $k_0 = 1$, $w = 1$, $T = 1$ and $h = 1$. You can see that there exists a critical value of the gain p_{crit} . The system (4.36) is stable for gains less then critical one and unstable for gains greater then critical.

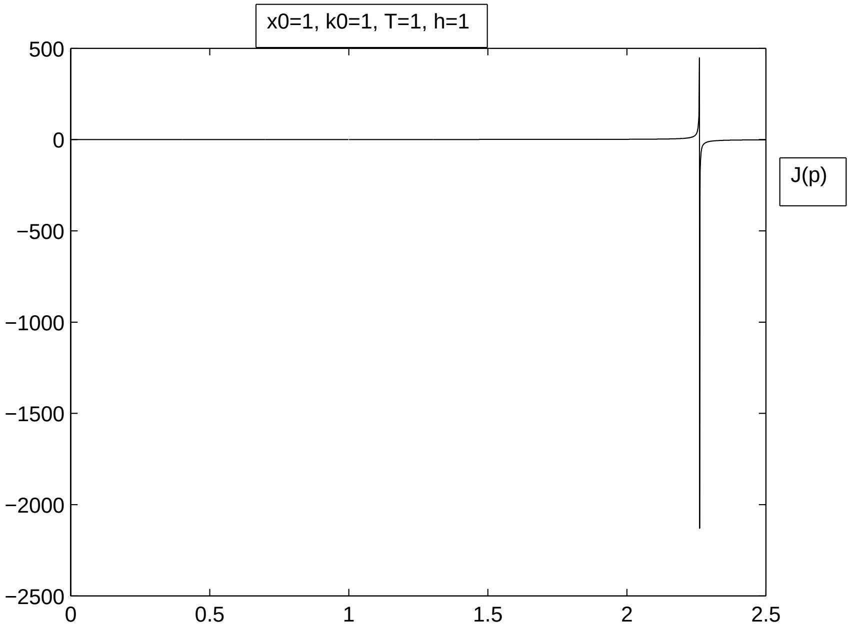

Fig. 4.1. Value of the index $J(p)$ for p greater then p_{crit}

Figure 4.2 shows the value of the index $J(p)$ for p less then critical gain. You can see that the function $J(p)$ is convex and has a minimum.

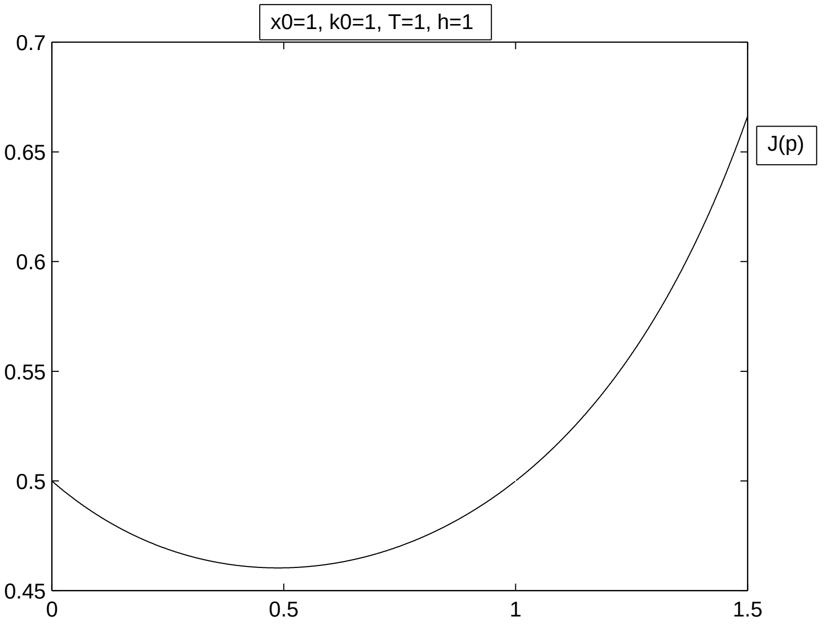

Fig. 4.2. Value of the index $J(p)$ for p less then p_{crit}

We search for an optimal gain which minimize the index (4.47) for a given $x_0 = 1$, $k_0 = 1$, $w = 1$ and $T = 1$. Optimization results, obtained by means of Matlab function *fminsearch*, are given in Table 4.1.

Table 4.1
Optimization results

Delay h	Optimal gain	Index value	Critical gain
0.1	7.10	0.13	16.350
0.2	3.50	0.22	8.502
0.5	1.25	0.37	3.806
1.0	0.50	0.46	2.261
2.0	0.14	0.495	1.519
3.0	0.05	0.499	1.292

4.5.2 Inertial system with delay and a PI-controller

Let us consider inertial system with time delay and a PI-controller [24]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{1}{T}x(t) + \dfrac{k_0}{T}u(t-h) \\ u(t) = -px(t) - \dfrac{1}{T_i}\displaystyle\int_0^t x(\xi)d\xi \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{4.48}$$

$t \geq 0$, $x(t) \in \mathbb{R}$, $\theta \in [-h, 0)$, k_0, T, T_i, $p \in \mathbb{R}$, $h \geq 0$. The parameter k_0 is a gain of a plant, p is a gain and T_i is a time of isodrome of a PI controller, T is a system time constant, x_0 – is the initial state. One introduces the state variables $x_1(t)$ and $x_2(t)$ as follows

$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = \dfrac{1}{T_i}\displaystyle\int_0^t x(\xi)d\xi \end{cases} \tag{4.49}$$

The set of equations (4.48) takes a form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -\dfrac{1}{T}x_1(t) + \dfrac{k_0}{T}u(t-r) \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{T_i}x_1(t) \\ x_1(0) = x_0 \\ x_2(0) = x_{20} \\ x_1(\theta) = 0 \\ x_2(\theta) = 0 \\ u(t) = -px_1(t) - x_2(t) \end{cases} \tag{4.50}$$

for $t \geq 0$, $\theta \in [-r, 0)$. One can reshape equation (4.50) to a form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -\dfrac{1}{T}x_1(t) - \dfrac{k_0 p}{T}x_1(t-h) - \dfrac{k_0}{T}x_2(t-h) \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{T_i}x_1(t) \\ x_1(0) = x_o \\ x_2(0) = x_{20} \\ x_1(\theta) = 0 \\ x_2(\theta) = 0 \end{cases} \tag{4.51}$$

for $t \geq 0$, $\theta \in [-r, 0)$.

Matrices

$$A_0 = \begin{bmatrix} -\frac{1}{T} & 0 \\ \frac{1}{T_i} & 0 \end{bmatrix} \tag{4.52}$$

$$A_1 = \begin{bmatrix} -\frac{k_0 p}{T} & -\frac{k_0}{T} \\ 0 & 0 \end{bmatrix} \tag{4.53}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_0^\infty \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt \tag{4.54}$$

where $w > 0$.
The value of the performance index of quality (4.54) is equal to the value of the Lyapunov functional for initial function of system (4.51).

$$J = \begin{bmatrix} x_0 & x_{20} \end{bmatrix} \begin{bmatrix} U_{11}(0) & U_{12}(0) \\ U_{21}(0) & U_{22}(0) \end{bmatrix} \begin{bmatrix} x_0 \\ x_{20} \end{bmatrix} \tag{4.55}$$

Where $U(\xi)$ for $\xi \in [0,h]$ is obtained by solving the set of equations (4.32), (4.27) and (4.28) which takes a form

$$\frac{d}{d\xi} \begin{bmatrix} U_{11}(\xi) \\ U_{21}(\xi) \\ U_{12}(\xi) \\ U_{22}(\xi) \\ Z_{11}(\xi) \\ Z_{21}(\xi) \\ Z_{12}(\xi) \\ Z_{22}(\xi) \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} U_{11}(\xi) \\ U_{21}(\xi) \\ U_{12}(\xi) \\ U_{22}(\xi) \\ Z_{11}(\xi) \\ Z_{21}(\xi) \\ Z_{12}(\xi) \\ Z_{22}(\xi) \end{bmatrix} \tag{4.56}$$

$$[I - \Phi_{21}(h)] \begin{bmatrix} U_{11}(0) \\ U_{21}(0) \\ U_{12}(0) \\ U_{22}(0) \end{bmatrix} - \Phi_{22}(h) \begin{bmatrix} Z_{11}(0) \\ Z_{21}(0) \\ Z_{12}(0) \\ Z_{22}(0) \end{bmatrix} = 0 \tag{4.57}$$

$$\Big[Q_{11} - Q_{22} - Q_{21}\Phi_{11}(h)\Big] \begin{bmatrix} U_{11}(0) \\ U_{21}(0) \\ U_{12}(0) \\ U_{22}(0) \end{bmatrix} + \Big[Q_{12} - Q_{21}\Phi_{12}(h)\Big] \begin{bmatrix} Z_{11}(0) \\ Z_{21}(0) \\ Z_{12}(0) \\ Z_{22}(0) \end{bmatrix} = \begin{bmatrix} -w \\ 0 \\ 0 \\ -w \end{bmatrix} \tag{4.58}$$

where

$$Q_{11} = \begin{bmatrix} -\frac{1}{T} & 0 & \frac{1}{T_i} & 0 \\ 0 & -\frac{1}{T} & 0 & \frac{1}{T_i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.59}$$

$$Q_{12} = \begin{bmatrix} -\frac{k_0 p}{T} & 0 & 0 & 0 \\ 0 & -\frac{k_0 p}{T} & 0 & 0 \\ -\frac{k_0}{T} & 0 & 0 & 0 \\ 0 & -\frac{k_0}{T} & 0 & 0 \end{bmatrix} \tag{4.60}$$

$$Q_{21} = \begin{bmatrix} \frac{k_0 p}{T} & 0 & 0 & 0 \\ \frac{k_0}{T} & 0 & 0 & 0 \\ 0 & 0 & \frac{k_0 p}{T} & 0 \\ 0 & 0 & \frac{k_0}{T} & 0 \end{bmatrix} \tag{4.61}$$

$$Q_{22} = \begin{bmatrix} \frac{1}{T} & -\frac{1}{T_i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{T} & -\frac{1}{T_i} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.62}$$

$$\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix} \tag{4.63}$$

$\Phi(\xi)$ is a fundamental matrix of system (4.56).

Problem 4.2. *Determine the parameters p and T_i whose minimize an integral quadratic performance index of quality (4.54).*

We search for an optimal parameters of a PI-controller which minimize the index (4.55). Optimization results, obtained by means of Matlab function *fminsearch*, are given in Table 4.2. These results are obtained for $x_0 = 1$, $x_{20} = 0.5$ $w = 1$, $T = 5$, and $k_0 = 1$.

Table 4.2
Optimization results

Delay h	Optimal p	Optimal $1/T_i$	Index value
1.0	3.7175	0.3693	7.0684
1.5	2.5023	0.2478	8.0224
2.0	1.9008	0.1877	8.9227
2.5	1.5442	0.1521	9.7723
3.0	1.3094	0.1287	10.5749
3.5	1.1442	0.1121	11.3346
4.0	1.0222	0.0997	12.0554

Figure 4.3 shows the value of the index $J(p)$ for fixed $1/T_i = 0.1877$ and $h = 2$. You can see that there exists a critical value of the gain p_{crit}. The system (4.51) is stable for gains less then critical one and unstable for gains greater then critical.

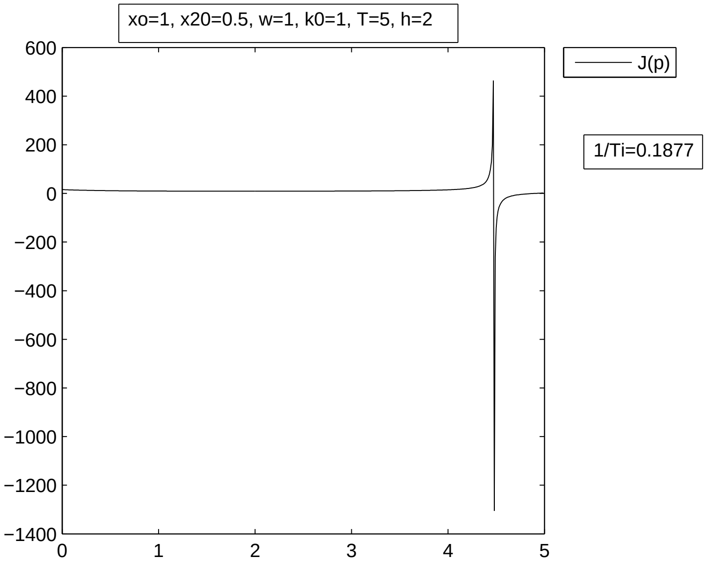

Fig. 4.3. Value of the index $J(p)$ for fixed $1/T_i = 0.1877$

Figure 4.4 shows the value of the index $J(p)$ for fixed $1/T_i = 0.1877$, $h = 2$ and gains less the critical one. You can see that the function $J(p)$ is convex and has a minimum. Figure 4.5 shows the value of the index $J(1/T_i)$ for fixed $p = 1.9008$ and $h = 2$. You can see that there exists a critical value of the parameter $1/T_{i_{crit}}$. The system (4.51) is stable for $1/T_i$ less then critical one and unstable for $1/T_i$ greater then critical.

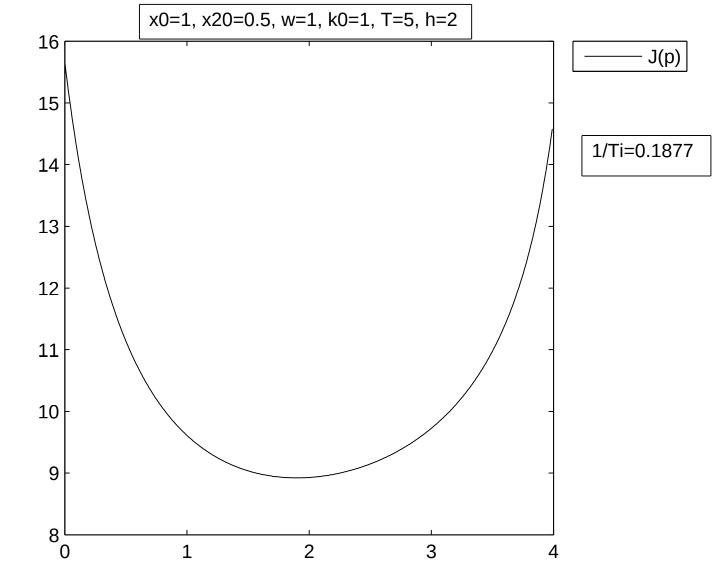

Fig. 4.4. Value of the index $J(p)$ for fixed $1/T_i = 0.1877$

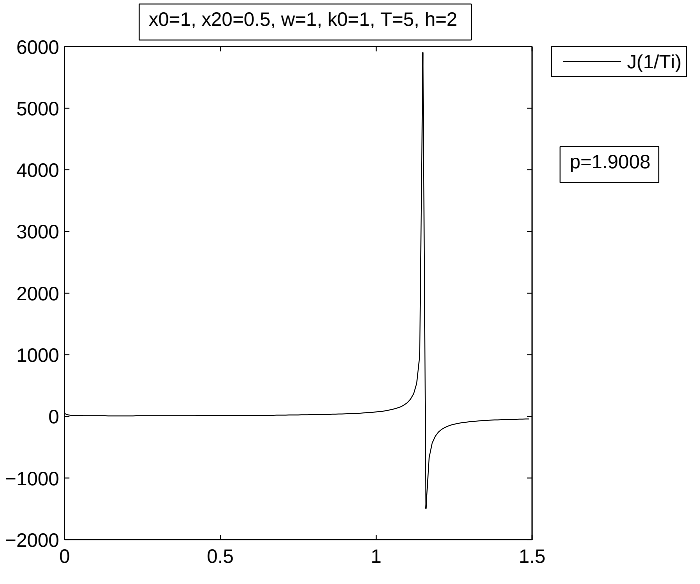

Fig. 4.5. Value of the index $J(1/T_i)$ for fixed $p = 1.9008$

Figure 4.6 shows the value of the index $J(1/T_i)$ for fixed $p = 1.9008$, $h = 2$ and $1/T_i$ less the critical one. You can see that the function $J(1/T_i)$ is convex and has a minimum.

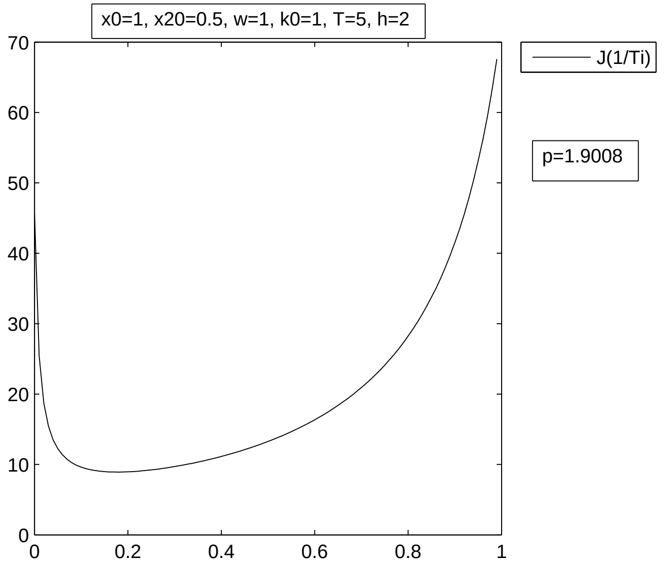

Fig. 4.6. Value of the index $J(1/T_i)$ for fixed $p = 1.9008$

Figure 4.7 shows elements of matrix $U(\xi)$ for optimal values of the PI controller parameters $p = 1.9008$ and $1/T_i = 0.1877$ for $h = 2$.

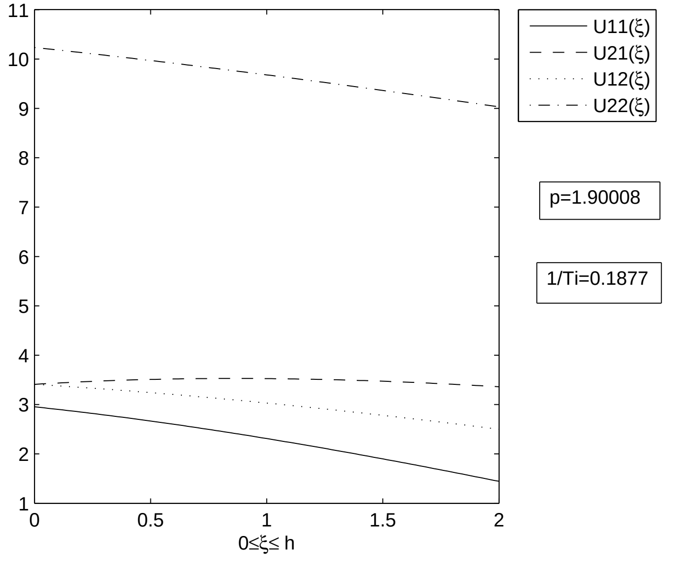

Fig. 4.7. Elements of matrix $U(\xi)$

4.6 The Lyapunov matrix for a system with two commensurate delays

Let us consider a system [25]

$$\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.64}$$

for $t \geq 0$ and $\theta \in [-2h, 0]$. Where $A_0, A_1, A_2 \in \mathbb{R}^{n \times n}$ and $\varphi \in PC([-h, 0], \mathbb{R}^n)$, $0 < h \in \mathbb{R}$. The set of equations (4.10), (4.11), (4.12) for system (4.64) takes a form

$$\frac{d}{d\xi} U(\xi) = U(\xi) A_0 + U(\xi - h) A_1 + U(\xi - 2h) A_2 \tag{4.65}$$

$$U(-\xi) = U^T(\xi) \tag{4.66}$$

$$U(0)A_0 + U(-h)A_1 + U(-2h)A_2 + A_0^T U(0) + A_1^T U(h) + A_2^T U(2h) = -W \tag{4.67}$$

for $\xi \in [0, 2h]$.
The relation (4.66) implies

$$U(-h) = U^T(h) \text{ and } U(-2h) = U^T(2h)$$

so we can write equation (4.67) in a form

$$U(0)A_0 + U^T(h)A_1 + U^T(2h)A_2 + A_0^T U(0) + A_1^T U(h) + A_2^T U(2h) = -W \tag{4.68}$$

Formula (4.66) extends the function U defined on the segment $[0, 2h]$ to the segment $[-2h, 0]$. Indeed for $\xi \in [0, 2h]$, $U(-\xi) = U^T(\xi)$. For $\tau = -\xi$, $U(\tau) = U^T(-\tau)$ and $\tau \in [-2h, 0]$.
We define the functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$ for $\xi \in [0, h]$

$$U_1(\xi) = U(\xi) \tag{4.69}$$

$$U_2(\xi) = U(h + \xi) \tag{4.70}$$

$$Z_1(\xi) = U(\xi - h) = U^T(-\xi + h) \tag{4.71}$$

$$Z_2(\xi) = U(\xi - 2h) = U^T(-\xi + 2h) \tag{4.72}$$

For $\xi \in [0,h]$ equation (4.65) can be written in a form

$$\frac{d}{d\xi}U_1(\xi) = U_1(\xi)A_0 + Z_1(\xi)A_1 + Z_2(\xi)A_2 \tag{4.73}$$

For $\xi + h = \varsigma \in [h,2h]$
$U(\varsigma) = U(\xi+h) = U_2(\xi),\, U(\varsigma-h) = U(\xi) = U_1(\xi),\, U(\varsigma-2h) = U(\xi-h) = Z_1(\xi)$
and equation (4.65) can be written in a form

$$\frac{d}{d\xi}U_2(\xi) = U_2(\xi)A_0 + U_1(\xi)A_1 + Z_1(\xi)A_2 \tag{4.74}$$

We compute the derivative of $Z_1(\xi)$

$$\begin{aligned}\frac{d}{d\xi}Z_1(\xi) &= \frac{d}{d\xi}U^T(-\xi+h) = \frac{d}{d\tau}U^T(\tau)\frac{d\tau}{d\xi} = -\frac{d}{d\tau}U^T(\tau) = \\ &= -A_0^T U^T(\tau) - A_1^T U^T(\tau-h) - A_2^T U^T(\tau-2h) = \\ &= -A_0^T U^T(-\xi+h) - A_1^T U^T(-\xi) - A_2^T U^T(-\xi-h) = \\ &= -A_0^T Z_1(\xi) - A_1^T U_1(\xi) - A_2^T U_2(\xi)\end{aligned} \tag{4.75}$$

where $\tau = -\xi + h$ and the derivative of $Z_2(\xi)$

$$\begin{aligned}\frac{d}{d\xi}Z_2(\xi) &= \frac{d}{d\xi}U^T(-\xi+2h) = \frac{d}{d\tau}U^T(\tau)\frac{d\tau}{d\xi} = -\frac{d}{d\tau}U^T(\tau) = \\ &= -A_0^T U^T(\tau) - A_1^T U^T(\tau-h) - A_2^T U^T(\tau-2h) = \\ &= -A_0^T U^T(-\xi+2h) - A_1^T U^T(-\xi+h) - A_2^T U^T(-\xi) = \\ &= -A_2^T U_1(\xi) - A_1^T Z_1(\xi) - A_0^T Z_2(\xi)\end{aligned} \tag{4.76}$$

where $\tau = -\xi + 2h$.
We have received the set of ordinary differential equations

$$\begin{cases} \frac{d}{d\xi}U_1(\xi) = U_1(\xi)A_0 + Z_1(\xi)A_1 + Z_2(\xi)A_2 \\ \frac{d}{d\xi}U_2(\xi) = U_1(\xi)A_1 + U_2(\xi)A_0 + Z_1(\xi)A_2 \\ \frac{d}{d\xi}Z_1(\xi) = -A_1^T U_1(\xi) - A_2^T U_2(\xi) - A_0^T Z_1(\xi) \\ \frac{d}{d\xi}Z_2(\xi) = -A_2^T U_1(\xi) - A_1^T Z_1(\xi) - A_0^T Z_2(\xi) \end{cases} \tag{4.77}$$

for $\xi \in [0,h]$ with initial conditions

$$U_1(0), U_2(0), Z_1(0), Z_2(0)$$

There hold relations

$$U(0) = U_1(0), U(h) = U_2(0), U(2h) = U_2(h)$$

and therefore equation (4.68) takes a form

$$U_1(0)A_0 + U_2^T(0)A_1 + U_2^T(h)A_2 + A_0^T U_1(0) + A_1^T U_2(0) + A_2^T U_2(h) = -W \tag{4.78}$$

Using the Kronecker product we can express equation (4.77) in a form

$$\begin{bmatrix} \frac{d}{d\xi} colU_1(\xi) \\ \frac{d}{d\xi} colU_2(\xi) \\ \frac{d}{d\xi} colZ_1(\xi) \\ \frac{d}{d\xi} colZ_2(\xi) \end{bmatrix} = \mathscr{H} \begin{bmatrix} colU_1(\xi) \\ colU_2(\xi) \\ colZ_1(\xi) \\ colZ_2(\xi) \end{bmatrix} \tag{4.79}$$

for $\xi \in [0, h]$ with initial conditions

$$colU_1(0), colU_2(0), colZ_1(0), colZ_2(0)$$

where

$$\mathscr{H} = \begin{bmatrix} A_0^T \otimes I & 0 & A_1^T \otimes I & A_2^T \otimes I \\ A_1^T \otimes I & A_0^T \otimes I & A_2^T \otimes I & 0 \\ -I \otimes A_1^T & -I \otimes A_2^T & -I \otimes A_0^T & 0 \\ -I \otimes A_2^T & 0 & -I \otimes A_1^T & -I \otimes A_0^T \end{bmatrix}$$

Formula (4.78) can be expressed in a form

$$(A_0^T \otimes I + I \otimes A_0^T) colU_1(0) + (I \otimes A_1^T) colU_2(0) +$$

$$+ (A_1^T \otimes I) colU_2^T(0) + (I \otimes A_2^T) colU_2(h) + (A_2^T \otimes I) colU_2^T(h) = -colW \tag{4.80}$$

Solution of equation (4.79) is given in a form

$$\begin{bmatrix} colU_1(\xi) \\ colU_2(\xi) \\ colZ_1(\xi) \\ colZ_2(\xi) \end{bmatrix} = \Phi(\xi) \begin{bmatrix} colU_1(0) \\ colU_2(0) \\ colZ_1(0) \\ colZ_2(0) \end{bmatrix} \tag{4.81}$$

where a matrix

$$\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) & \Phi_{23}(\xi) & \Phi_{24}(\xi) \\ \Phi_{31}(\xi) & \Phi_{32}(\xi) & \Phi_{33}(\xi) & \Phi_{34}(\xi) \\ \Phi_{41}(\xi) & \Phi_{42}(\xi) & \Phi_{43}(\xi) & \Phi_{44}(\xi) \end{bmatrix} \tag{4.82}$$

is a fundamental matrix of system (4.79).
We determine the initial conditions

$$colU_1(0), colU_2(0), colZ_1(0), colZ_2(0)$$

From equation (4.81) we obtain

$$\begin{aligned} colU_1(h) = colU_2(0) = \Phi_{11}(h)colU_1(0) + \Phi_{12}(h)colU_2(0) + \\ + \Phi_{13}(h)colZ_1(0) + \Phi_{14}(h)colZ_2(0) \end{aligned} \tag{4.83}$$

$$\begin{aligned} colZ_1(h) = colU_1(0) = \Phi_{31}(h)colU_1(0) + \Phi_{32}(h)colU_2(0) + \\ + \Phi_{33}(h)colZ_1(0) + \Phi_{34}(h)colZ_2(0) \end{aligned} \tag{4.84}$$

$$\begin{aligned} colZ_2(h) = colZ_1(0) = \Phi_{41}(h)colU_1(0) + \Phi_{42}(h)colU_2(0) + \\ + \Phi_{43}(h)colZ_1(0) + \Phi_{44}(h)colZ_2(0) \end{aligned} \tag{4.85}$$

$$\begin{aligned} colU_2(h) = \Phi_{21}(h)colU_1(0) + \Phi_{22}(h)colU_2(0) + \Phi_{23}(h)colZ_1(0) + \\ + \Phi_{24}(h)colZ_2(0) \end{aligned} \tag{4.86}$$

We reshape equations (4.83), (4.84) and (4.85). In this way we attain a set of algebraic equations which enables us to calculate the initial conditions of system (4.79).

$$\Phi_{11}(h)colU_1(0) + (\Phi_{12}(h) - 1)colU_2(0) + \Phi_{13}(h)colZ_1(0) + \Phi_{14}(h)colZ_2(0) = 0 \tag{4.87}$$

$$(\Phi_{31}(h) - 1)colU_1(0) + \Phi_{32}(h)colU_2(0) + \Phi_{33}(h)colZ_1(0) + \Phi_{34}(h)colZ_2(0) = 0 \tag{4.88}$$

$$\Phi_{41}(h)colU_1(0) + \Phi_{42}(h)colU_2(0) + (\Phi_{43}(h) - 1)colZ_1(0) + \Phi_{44}(h)colZ_2(0) = 0 \tag{4.89}$$

$$colU_2(h) = \Phi_{21}(h)colU_1(0) + \Phi_{22}(h)colU_2(0) + \Phi_{23}(h)colZ_1(0) + \Phi_{24}(h)colZ_2(0) \tag{4.90}$$

$$\begin{aligned} (A_0^T \otimes I + I \otimes A_0^T)colU_1(0) + (I \otimes A_1^T)colU_2(0) + (A_1^T \otimes I)colU_2^T(0) + \\ + (I \otimes A_2^T)colU_2(h) + (A_2^T \otimes I)colU_2^T(h) = -colW \end{aligned} \tag{4.91}$$

4.7 Formulation of the parametric optimization problem

Let us consider a time-delay system with a P-controller

$$\begin{cases} \dfrac{dx(t)}{dt} = \sum\limits_{j=0}^{m} A_j x(t-h_j) + Bu(t-h) \\ u(t) = -Px(t) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{4.92}$$

for $t \geq t_0,\ \theta \in [-h,0]$

Where $x(t) \in \mathbb{R}^n$ is the state of system (4.92), $u(t) \in \mathbb{R}^p$ is the control, $A_j \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $P \in \mathbb{R}^{p\times n}$ is a P-controller gain, $\varphi \in PC([-h,0],\mathbb{R}^n)$ is the initial function, $0 = h_0 < h_1 < \ldots < h_m = h$ are delays.

System (4.92) can be written in an equivalent form

$$\begin{cases} \dfrac{dx(t)}{dt} = \sum\limits_{j=0}^{m} A_j x(t-h_j) - BPx(t-h) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{4.93}$$

for $t \geq t_0,\ \theta \in [-h,0]$.

In parametric optimization problem will be used the performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t,t_0,\varphi) W x(t,t_0,\varphi) dt \tag{4.94}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,t_0,\varphi)$ is a solution of equation (4.93) for initial function φ.

Problem 4.3. *Determine the matrix $P \in \mathbb{R}^{p\times n}$ whose minimize an integral quadratic performance index of quality (4.94)*

According to equation (4.5) the value of the performance index of quality (4.94) is equal to the value of the functional (4.9) for initial function φ. To calculate the value of the functional (4.9) we need a Lyapunov matrix $U(\xi)$.
To obtain a Lyapunov matrix $U(\xi)$ we have to solve a system of equations (4.10), (4.11) and (4.12).

4.8 The example. Parametric optimization problem for a scalar system with two delays

Let us consider a system with two delays and a P-controller [25]

$$\begin{cases} \dfrac{dx(t)}{dt} = ax(t) + bx(t-h) + cx(t-2h) + u(t-2h) \\ u(t) = -px(t) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.95}$$

$t \geq 0$, $x(t) \in \mathbb{R}$ is the state of system (4.95), $u(t) \in \mathbb{R}$ is the control, $\varphi(\theta)$ for $\theta \in [-2h, 0]$ is the initial function, $0 \leq h$, $2h$ are time delays, the parameter p is a gain of a P-controller. One can reshape equation (4.95) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} = ax(t) + bx(t-h) + (c-p)x(t-2h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{4.96}$$

for $t \geq 0$ and $\theta \in [-2h, 0]$.
In parametric optimization problem we use the performance index of quality

$$J = \int_0^\infty wx^2(t,\varphi)dt \tag{4.97}$$

where $w > 0$ and $x(t,\varphi)$ is a solution of equation (4.96) for initial function φ.
The Lyapunov functional for system (4.96) has a form, see formula (4.9)

$$\begin{aligned} v(\varphi) = {} & U(0)\varphi^2(0) + 2b\varphi(0)\int_{-h}^{0} U(-\theta-h)\varphi(\theta)d\theta + \\ & +2(c-p)\varphi(0)\int_{-h}^{0} U(-\theta-2h)\varphi(\theta)d\theta + b^2\int_{-h}^{0}\int_{-h}^{0} U(\theta-\eta)\varphi(\theta)\varphi(\eta)d\eta d\theta + \\ & +2b(c-p)\int_{-h}^{0}\int_{-2h}^{0} U(-h+\theta-\eta)\varphi(\theta)\varphi(\eta)d\eta d\theta + \\ & +(c-p)^2\int_{-2h}^{0}\int_{-2h}^{0} U(\theta-\eta)\varphi(\theta)\varphi(\eta)d\eta d\theta \end{aligned} \tag{4.98}$$

The value of the performance index of quality (4.97) is equal to the value of the functional (4.98) for initial function φ

$$J = v(\varphi) \tag{4.99}$$

To obtain the value of the performance index of quality one needs a Lyapunov matrix $U(\xi)$ for $\xi \in [0,2h]$. In Chapter 3.6 was presented a method of determination of the Lyapunov matrix for a system with two delays.
System of equations (4.77) takes a form

$$\begin{bmatrix} \frac{d}{d\xi}U_1(\xi) \\ \frac{d}{d\xi}U_2(\xi) \\ \frac{d}{d\xi}Z_1(\xi) \\ \frac{d}{d\xi}Z_2(\xi) \end{bmatrix} = G \begin{bmatrix} U_1(\xi) \\ U_2(\xi) \\ Z_1(\xi) \\ Z_2(\xi) \end{bmatrix} \tag{4.100}$$

where

$$G = \begin{bmatrix} a & 0 & b & c-p \\ b & a & c-p & 0 \\ -b & -c+p & -a & 0 \\ -c+p & 0 & -b & -a \end{bmatrix}$$

Initial conditions of system (4.100) one obtains solving the algebraic equation

$$Q \begin{bmatrix} U_1(0) \\ U_2(0) \\ Z_1(0) \\ Z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -w \end{bmatrix} \tag{4.101}$$

where

$$Q = \begin{bmatrix} \Phi_{11}(h) & \Phi_{12}(h)-1 & \Phi_{13}(h) & \Phi_{14}(h) \\ \Phi_{31}(h)-1 & \Phi_{32}(h) & \Phi_{33}(h) & \Phi_{34}(h) \\ \Phi_{41}(h) & \Phi_{42}(h) & \Phi_{43}(h)-1 & \Phi_{44}(h) \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \tag{4.102}$$

$p_{41} = 2a+2(c-k)\Phi_{21}(h)$, $p_{42} = 2b+2(c-k)\Phi_{22}(h)$, $p_{43} = 2(c-k)\Phi_{23}(h)$, $p_{44} = 2(c-k)\Phi_{24}(h)$, $\Phi(\xi)$ is a fundamental matrix of solutions of equation (4.100).
We search for an optimal gain which minimizes the index (4.97) for the initial function φ given by the formula

$$\varphi(\theta) = \begin{cases} x_0 & \text{for } \theta = 0 \\ 0 & \text{for } \theta \in [-2h,0) \end{cases} \tag{4.103}$$

The value of functional (4.98) for φ given by formula (4.103) is equal to

$$J(p) = v(\varphi) = U(0)x_0^2 \tag{4.104}$$

Figure 4.8 shows the value of the index $J(p)$ for $a = -1$, $b = -0.5$, $c = 1$ and $h = 1$. You can see that there exists a critical value of the gain p_{crit}. The system (4.96) is stable for gains less then critical one and unstable for gains greater then critical.

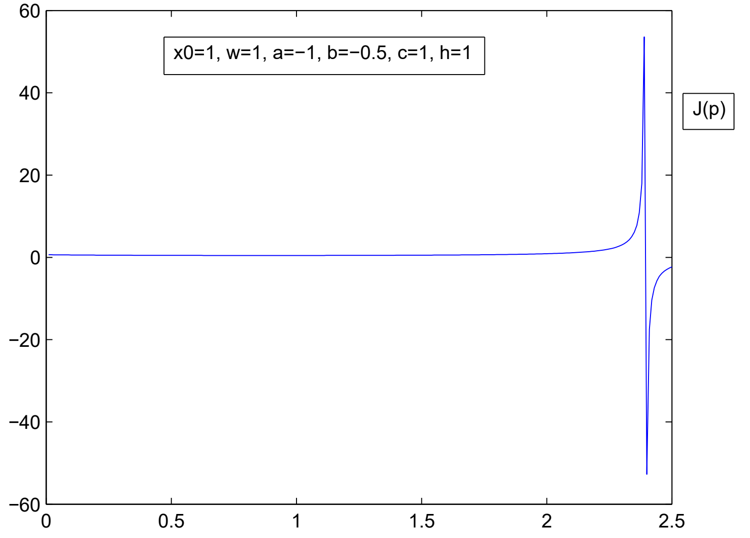

Fig. 4.8. Value of the index $J(p)$

Figure 4.9 shows the value of the index $J(p)$ for $a = -1$, $b = -0.5$, $c = 1$, $h = 1$ and for p less then critical gain. You can see that the function $J(p)$ is convex and has a minimum.

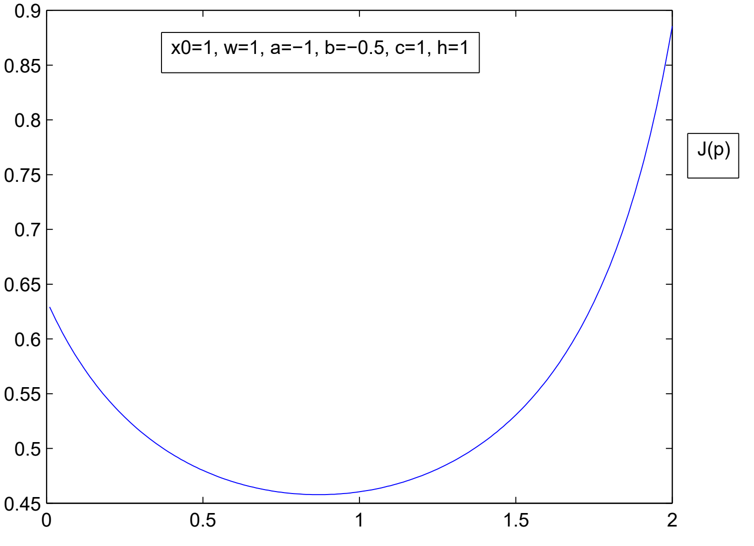

Fig. 4.9. Value of the index $J(p)$

Optimization results, obtained by means of Matlab function *fminsearch* for $a = -1$, $b = -0.5$, $c = 1$, $x_0 = 1$, are given in Table 4.3.

Table 4.3
Optimization results

Delay h	Optimal gain	Critical gain	Index value
0.5	1.15	3.13	0.4043
1.0	0.87	2.39	0.4578
1.5	0.90	2.17	0.4964
2.0	0.96	2.08	0.5252
2.5	1.02	2.03	0.5428

Figure 4.10 shows graphs of functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$ and $Z_2(\xi)$ obtained with the Matlab code, for parameters of system (4.96) used in optimization process with $h = 1$ and for optimal gain $p = 0.87$.

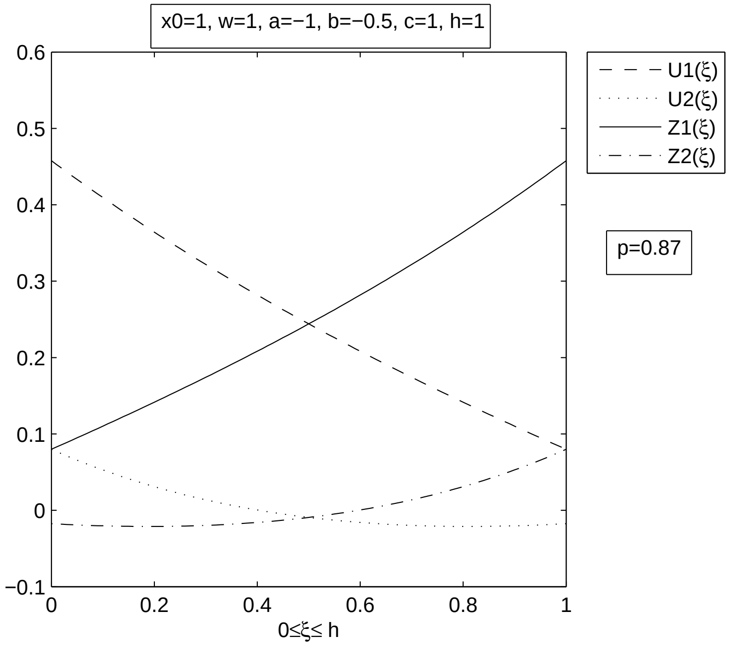

Fig. 4.10. Functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$ and $Z_2(\xi)$ for optimal gain $p = 0.87$

5 The Lyapunov matrix for a neutral system

5.1 The Lyapunov matrix for a neutral system with one delay

5.1.1 Mathematical model of a neutral system with one delay

Let us consider a neutral system

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-r)}{dt} = Ax(t) + Bx(t-r) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{5.1}$$

for $t \geq t_0,\ \theta \in [-r,0],\ r > 0$

Where $x(t) \in \mathbb{R}^n$, A, B, $C \in \mathbb{R}^{n\times n}$, function $\varphi \in PC^1([-r,0],\mathbb{R}^n)$ - is a space of piece-wise continuously differentiable vector valued functions defined on the segment $[-r,0]$ with the uniform norm $\| \varphi \|_{PC^1} = \sup\limits_{\theta\in[-r,0]} \| \varphi(\theta) \|$

Let $x(t,t_0,\varphi)$ be the solution of system (5.1) with the initial function φ for $t \geq t_0$.

Definition 5.1. *The* ***difference equation*** *associated with (5.1) is given by a term*

$$x(t) - Cx(t-r) = 0 \tag{5.2}$$

for $t \geq t_0$

We assume that the difference $x(t) - Cx(t-r)$ is continuous and differentiable for $t \geq t_0$, except possibly a countable number of points.

Let $x(t,\varphi)$ be the solution of system (5.1) with the initial function φ for $t \geq t_0$.

The initial condition for equation (5.1) can be written in a form

$$x_{t_0}(\varphi) = \varphi \tag{5.3}$$

where $x_t(\varphi) \in PC^1([-r,0],\mathbb{R}^n)$ is a shifted restriction of the function $x(\cdot,t_0,\varphi)$ to the segment $[-r,0]$.

The eigenvalues of the neutral equation (5.1) for large modulus are asymptotically equal to the eigenvalues of the difference equation (5.2).
According to the Theorem 9.6.1 [40] the difference equation (5.2) is stable when the matrix C is Schur stable.
When the matrix C is Schur stable, then the asymptotic stability of system (5.1) is equivalent to the exponential stability of the system (5.1). We assume that C is not singular and a Schur stable matrix.

Definition 5.2. [2] *$K(t)$ is the* ***fundamental matrix*** *of system (5.1) if it satisfies the matrix equation*

$$\frac{d}{dt}K(t) - C\frac{d}{dt}K(t-r) = AK(t) + BK(t-r)$$

for $t \geq 0$ and the following conditions

- *initial condition*: $K(0) = I_{n\times n}$ and $K(t) = 0_{n\times n}$ for $t < 0$ where $I_{n\times n}$ is the identity $n \times n$ matrix and $0_{n\times n}$ is the zero $n \times n$ matrix,
- *sewing condition*: $K(t) - CK(t-r)$ is continuous for $t > 0$.

It follows from the definition that the fundamental matrix $K(t)$ has discontinuity points.
The sewing condition implies *the jump equation*

$$\triangle K(t) - C\triangle K(t-r) = 0 \tag{5.4}$$

for $t \geq 0$, where $\triangle K(t) = K(t+0) - K(t-0)$
To compute the size of the jumps one needs to solve the jump equation (5.4) at $t_j = jr$, $j = 0, 1, 2, ...$, with the initial condition $\triangle K(0) = I$.

Lemma 5.1. [2] *The fundamental matrix $K(t)$ has jumps at points $t_j = jr$, $j = 0, 1, 2, ...$*

$$\triangle K(t) \mid_{t=t_j} = K(jr+0) - K(jr-0) = C^j \tag{5.5}$$

and $K(t) = K(t+0)$ at the jump points.

Theorem 5.1. [2] *Let $K(t)$ be the fundamental matrix of system (5.1), then for $t \geq t_0$*

$$x(t,\varphi) = [K(t-t_0) - K(t-t_0-r)]\,\varphi(0) + \\ + \int_{-r}^{0} K(t-t_0-r-\theta)\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]d\theta \tag{5.6}$$

This expression is called *the Cauchy formula* for system (5.1).

Theorem 5.2. [2] *The fundamental matrix $K(t)$ of system (5.1) satisfies also the equation*

$$\frac{d}{dt}K(t) - \frac{d}{dt}K(t-r)C = K(t)A + K(t-r)B \tag{5.7}$$

for $t > 0$ and $t \neq jr$, $j = 1, 2, ...$

5.1.2 The Lyapunov–Krasovskii functional for a neutral system with one delay

Given a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$. We are looking for a functional

$$v : PC^1([-r,0],\mathbb{R}^n) \to \mathbb{R}$$

such that along the solutions of system (5.1) the following equality holds

$$\frac{d}{dt}v(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi) \tag{5.8}$$

for $t \geq t_0$, where $x(t,\varphi)$ is a solution of system (5.1), with the initial function $\varphi \in PC^1([-r,0],\mathbb{R}^n)$, given by (5.6) and $x_t(\varphi)$ is a shifted restriction of $x(\cdot,\varphi)$ to an interval $[t-r,t]$.

We assume that system (5.1) is asymptotically stable and integrate both sides of equation (5.8) from t_0 to infinity. We obtain

$$v(x_{t_0}(\varphi)) = \int_{t_0}^{\infty} x^T(t,\varphi)Wx(t,\varphi)dt \tag{5.9}$$

Taking into account (5.6) we calculate the integral of the right-hand side of equation (5.9)

$$\begin{aligned}
\int_{t_0}^{\infty} x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt &= \varphi^T(0)\int_0^{\infty} K^T(t)WK(t)dt\varphi(0)+ \\
&-\varphi^T(0)\int_0^{\infty} K^T(t)WK(t-r)dtC\varphi(0)+ \\
&-\varphi^T(0)C^T\int_0^{\infty} K^T(t-r)WK(t)dt\varphi(0)+\varphi^T(0)C^T\int_0^{\infty} K^T(t-r)WK(t-r)dtC\varphi(0)+ \\
&+2\varphi^T(0)\int_{-r}^{0}\left[\int_0^{\infty} K^T(t)WK(t-r-\theta)dt\right]\left[B\varphi(\theta)+C\frac{d}{d\theta}\varphi(\theta)\right]d\theta+ \\
&-2\varphi^T(0)C^T\int_{-r}^{0}\left[\int_0^{\infty} K^T(t-r)WK(t-r-\theta)dt\right]\left[B\varphi(\theta)+C\frac{d}{d\theta}\varphi(\theta)\right]d\theta+ \\
&+\int_{-r}^{0}\int_{-r}^{0}\left[\varphi^T(\theta)B^T+\frac{d}{d\theta}\varphi^T(\theta)C^T\right]\left[\int_0^{\infty} K^T(t-r-\theta)WK(t-r-\xi)dt\right]\times \\
&\times\left[B\varphi(\xi)+C\frac{d}{d\xi}\varphi(\xi)\right]d\theta d\xi
\end{aligned} \tag{5.10}$$

Using the Lyapunov matrix $U(\xi)$ (4.8) we attain a formula for the functional $v(x_{t_0}(\varphi))$

$$v(x_{t_0}(\varphi)) = \varphi^T(0)[U(0) - U(-r)C - C^T U^T(-r) + C^T U(0)C]\varphi(0) +$$
$$+2\varphi^T(0)\int_{-r}^{0}\left[U(-\theta - r) - C^T U(-\theta)\right]\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]d\theta +$$
$$+\int_{-r}^{0}\int_{-r}^{0}\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]^T U(\theta - \xi)\left[B\varphi(\xi) + C\frac{d}{d\xi}\varphi(\xi)\right]d\theta d\xi \quad (5.11)$$

Lemma 5.2. [81] *Let system (5.1) be exponentially stable. Then for every symmetric matrix $W \in \mathbb{R}^{n\times n}$, matrix $U(\xi)$ is well defined and satisfies the following properties:*
Dynamic property

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi - r)C = U(\xi)A + U(\xi - r)B \quad (5.12)$$

for $\xi \geq 0$ and $\xi \neq jr,\ j = 0,1,2,\ldots$
Symmetry property

$$U(-\xi) = U^T(\xi) \quad (5.13)$$

for $\xi \geq 0$
Algebraic property

$$-W = A^T U(0) + U(0)A - A^T U(-r)C - C^T U^T(-r)A +$$
$$+B^T U^T(-r) + U(-r)B - B^T U(0)C - C^T U(0)B \quad (5.14)$$

Using the symmetry property one can express the formula (5.11) in a form

$$v(x_{t_0}(\varphi)) = \varphi^T(0)[U(0) - U^T(r)C - C^T U(r) + C^T U(0)C]\varphi(0) +$$
$$+2\varphi^T(0)\int_{-r}^{0}\left[U(\theta + r) - U(\theta)C\right]^T\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]d\theta +$$
$$+\int_{-r}^{0}\int_{-r}^{0}\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]^T U(\theta - \xi)\left[B\varphi(\xi) + C\frac{d}{d\xi}\varphi(\xi)\right]d\theta d\xi \quad (5.15)$$

Using equation (5.3) one can express a relation (5.15) more general in a form

$$v(x_{t_0}(\varphi)) = x_{t_0}(\varphi)^T(0)[U(0) - U^T(r)C - C^T U(r) + C^T U(0)C]x_{t_0}(\varphi)(0) +$$
$$+2x_{t_0}^T(\varphi)(0)\int_{-r}^{0}\left[U(\theta + r) - U(\theta)C\right]^T\left[Bx_{t_0}(\varphi)(\theta) + C\frac{d}{d\theta}x_{t_0}(\varphi)(\theta)\right]d\theta +$$
$$+\int_{-r}^{0}\int_{-r}^{0}\left[Bx_{t_0}(\varphi)(\theta) + C\frac{d}{d\theta}x_{t_0}(\varphi)(\theta)\right]^T U(\theta - \xi)\left[Bx_{t_0}(\varphi)(\xi) + C\frac{d}{d\xi}x_{t_0}(\varphi)(\xi)\right]d\theta d\xi$$
$$(5.16)$$

Lemma 5.3. [81] *The Lyapunov matrix $U(\xi)$ for system (5.1) is continuously differentiable at $\xi \neq jr$, $j = 0, 1, 2, ...$, and at $\xi = jr$ matrix $dU(\xi)/d\xi$ has the jump*

$$\frac{d}{d\xi}U(jr+0) - \frac{d}{d\xi}U(jr-0) = -(Q-W)C^j \tag{5.17}$$

Here Q is the solution of the matrix equation

$$Q - C^T QC = W \tag{5.18}$$

5.1.3 The Lyapunov matrix for a neutral system with one delay

To obtain a Lyapunov matrix for a neutral system one needs to solve the set of equations [22]

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi - r)C = U(\xi)A + U(\xi - r)B \tag{5.19}$$

$$U(-\xi) = U^T(\xi) \tag{5.20}$$

$$\begin{aligned} -W = A^T U(0) + U(0)A - A^T U(-r)C - C^T U^T(-r)A + \\ + B^T U^T(-r) + U(-r)B - B^T U(0)C - C^T U(0)B \end{aligned} \tag{5.21}$$

Formula (5.20) implies

$$U(\xi - r) = U^T(-\xi + r) \tag{5.22}$$

and equation (5.19) takes a form

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U^T(-\xi + r)C = U(\xi)A + U^T(-\xi + r)B \tag{5.23}$$

We introduce a new variable $\tau = -\xi + r$. The term (5.23) for a new variable has a form

$$\frac{d}{d\tau}U^T(-\tau + r) - C^T \frac{d}{d\tau}U(\tau) = -A^T U^T(-\tau + r) - B^T U(\tau) \tag{5.24}$$

One obtains the set of equations

$$\begin{cases} \dfrac{d}{d\xi}U(\xi) - \dfrac{d}{d\xi}U^T(-\xi + r)C = U(\xi)A + U^T(-\xi + r)B \\ \dfrac{d}{d\xi}U^T(-\xi + r) - C^T \dfrac{d}{d\xi}U(\xi) = -A^T U^T(-\xi + r) - B^T U(\xi) \end{cases} \tag{5.25}$$

We introduce a new function

$$Z(\xi) = U^T(-\xi + r) \tag{5.26}$$

The set of equations (5.25) can be written in a form

$$\begin{cases} \frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}Z(\xi)C = U(\xi)A + Z(\xi)B \\ \frac{d}{d\xi}Z(\xi) - C^T\frac{d}{d\xi}U(\xi) = -A^TZ(\xi) - B^TU(\xi) \end{cases} \tag{5.27}$$

or in a equivalent form

$$\begin{cases} \frac{d}{d\xi}U(\xi) - C^T\frac{d}{d\xi}U(\xi)C = U(\xi)A - B^TU(\xi)C + Z(\xi)B - A^TZ(\xi)C \\ \frac{d}{d\xi}Z(\xi) - C^T\frac{d}{d\xi}Z(\xi)C = -B^TU(\xi) + C^TU(\xi)A - A^TZ(\xi) + C^TZ(\xi)B \end{cases} \tag{5.28}$$

for $\xi \in [0,r]$ with the initial conditions $U(0)$ and $Z(0)$.
The formulas (5.20) and (5.26) imply

$$U(-r) = U^T(r) = Z(0) \tag{5.29}$$

Taking into account (5.29) one can write the algebraic property (5.21) in a form

$$\begin{aligned} -W &= A^TU(0) + U(0)A - A^TZ(0)C - C^TZ^T(0)A + \\ &+ B^TZ^T(0) + Z(0)B - B^TU(0)C - C^TU(0)B \end{aligned} \tag{5.30}$$

Equation (5.28) can be written in a form

$$\begin{bmatrix} \frac{d}{d\xi}colU(\xi) \\ \frac{d}{d\xi}colZ(\xi) \end{bmatrix} = \mathscr{H} \begin{bmatrix} colU(\xi) \\ colZ(\xi) \end{bmatrix} \tag{5.31}$$

Solution of equation (5.31) is given by the formula

$$\begin{bmatrix} colU(\xi) \\ colZ(\xi) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix} \begin{bmatrix} colU(0) \\ colZ(0) \end{bmatrix} \tag{5.32}$$

where a matrix $\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix}$ is a fundamental matrix of system (5.31).
We determine the initial conditions $colU(0)$, $colZ(0)$.
The term (5.26) implies $Z(r) = U^T(0) = U(0)$.
From equation (5.32) we obtain

$$colZ(r) = colU(0) = \Phi_{21}(r)colU(0) + \Phi_{22}(r)colZ(0) \tag{5.33}$$

In this way we attain the set of algebraic equations which enables us to calculate the initial conditions of equation (5.32).

$$A^T U(0) + U(0)A - A^T Z(0)C - C^T Z^T(0)A + B^T Z^T(0) + \\ + Z(0)B - B^T U(0)C - C^T U(0)B = -W \tag{5.34}$$

$$[\Phi_{21}(r) - I] colU(0) + \Phi_{22}(r) colZ(0) = 0 \tag{5.35}$$

5.1.4 Formulation of the parametric optimization problem for a neutral system with one delay

Let us consider a neutral system with a P-controller [22]

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-r)}{dt} = Ax(t) + Bu(t-r) \\ u(t) = -Px(t) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{5.36}$$

for $t \geq t_0$, $\theta \in [-r, 0]$
Where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $A, C \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $P \in \mathbb{R}^{p\times n}$ is a P-controller gain, $\varphi \in PC([-r, 0], \mathbb{R}^n)$.
System (5.36) can be written in an equivalent form

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-r)}{dt} = Ax(t) - BPx(t-r) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{5.37}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t,\varphi) W x(t,\varphi) dt \tag{5.38}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,\varphi)$ is a solution of equation (5.37) for initial function φ.

Problem 5.1. *Determine the matrix $P \in \mathbb{R}^{p\times n}$ whose minimize an integral quadratic performance index of quality (5.38).*

According to equation (5.9) the value of the performance index of quality (5.38) is equal to the value of the functional (5.16) for initial function φ. To calculate the value of the functional (5.16) we need a Lyapunov matrix $U(\xi)$. To obtain a Lyapunov matrix $U(\xi)$ we have to solve a system of equations (5.12)–(5.14).

5.1.5 The examples

5.1.5.1 A linear neutral system with a P-controller

Let us consider a neutral system with a P-controller [22]

$$\begin{cases} \dfrac{dx(t)}{dt} - c\dfrac{dx(t-r)}{dt} = ax(t) + bu(t-r) \\ u(t) = -px(t) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{5.39}$$

$t \geq 0$, $x(t)$, $u(t) \in \mathbb{R}$, $\theta \in [-r, 0]$, $r \geq 0$. The parameter p is a gain of a P-controller, $x_0 \in \mathbb{R}$ is an initial state of system.
One can reshape equation (5.39) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} - c\dfrac{dx(t-r)}{dt} = ax(t) - bpx(t-r) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{5.40}$$

for $t \geq 0$ and $\theta \in [-r, 0)$.
The initial function φ is given by a term

$$\varphi(\theta) = \begin{cases} x_0 & \text{for } \theta = 0 \\ 0 & \text{for } \theta \in [-r, 0) \end{cases} \tag{5.41}$$

In parametric optimization problem we use the performance index

$$J = \int_0^\infty wx^2(t, \varphi)dt \tag{5.42}$$

where $w > 0$ and $x(t, \varphi)$ is a solution of (5.40) for initial function (5.41).
System of equations (5.31) takes a form

$$\begin{bmatrix} \dfrac{d}{d\xi}U(\xi) \\ \\ \dfrac{d}{d\xi}Z(\xi) \end{bmatrix} = \begin{bmatrix} \dfrac{a+bcp}{1-c^2} & -\dfrac{ac+bp}{1-c^2} \\ \\ \dfrac{ac+bp}{1-c^2} & -\dfrac{a+bcp}{1-c^2} \end{bmatrix} \begin{bmatrix} U(\xi) \\ \\ Z(\xi) \end{bmatrix} \tag{5.43}$$

A fundamental matrix of solutions of equation (5.43) has a form

$$\Phi(\xi)=\begin{bmatrix} \cosh\lambda\xi+\dfrac{a+bcp}{\lambda(1-c^2)}\sinh\lambda\xi & -\dfrac{ac+bp}{\lambda(1-c^2)}\sinh\lambda\xi \\ \\ \dfrac{ac+bp}{\lambda(1-c^2)}\sinh\lambda\xi & \cosh\lambda\xi-\dfrac{a+bcp}{\lambda(1-c^2)}\sinh\lambda\xi \end{bmatrix} \tag{5.44}$$

where

$$\lambda=\sqrt{\frac{a^2-b^2p^2}{1-c^2}} \tag{5.45}$$

Initial conditions of system (5.43) one obtains solving of the algebraic equation

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\begin{bmatrix} U(0) \\ Z(0) \end{bmatrix}=\begin{bmatrix} -w \\ 0 \end{bmatrix} \tag{5.46}$$

where

$$\begin{aligned} q_{11}&=2(a+bcp)\\ q_{12}&=-2(ac+bp)\\ q_{21}&=\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r-1\\ q_{22}&=\cosh\lambda r-\frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r \end{aligned}$$

Solving equation (5.46) we obtain

$$U(0)=\frac{w}{M}\left[-\cosh\lambda r+\frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r\right] \tag{5.47}$$

$$Z(0)=\frac{w}{M}\left[\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r-1\right] \tag{5.48}$$

where

$$M=2(a+bcp)\cosh\lambda r-2\lambda(1-c^2)\sinh\lambda r-2(ac+bp) \tag{5.49}$$

Solution of equation (5.43) has a form

$$U(\xi)=\frac{w}{M}\left[-\cosh\lambda r+\frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r\right]\cosh\lambda\xi-\frac{w}{2\lambda(1-c^2)}\sinh\lambda\xi \tag{5.50}$$

$$\begin{aligned} Z(\xi)=&\frac{w}{M}\left[\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r-1\right]\cosh\lambda\xi+\\ &+\frac{w}{M\lambda(1-c^2)}\left[a+bcp-(ac+bp)\cosh\lambda r\right]\sinh\lambda\xi \end{aligned} \tag{5.51}$$

The value of the performance index (5.42) is equal to the value of the functional (5.16) for initial function. In this example initial function is given by (5.41).

$$J = x_0^2 \left[(1+c^2)U(0) - 2cZ(0) \right] \tag{5.52}$$

After calculations one obtains

$$J = \frac{\dfrac{wx_0^2}{2}\left(2c - (1+c^2)\cosh\lambda r + \dfrac{a-bcp}{\lambda}\sinh\lambda r\right)}{-ac - bp + (a+bcp)\cosh\lambda r - \lambda(1-c^2)\sinh\lambda r} \tag{5.53}$$

Figure 5.1 shows the value of the index $J(p)$ for $x_0 = 1$, $w = 1$, $a = -1$, $b = 0.5$, $c = -0.6$ and $r = 1$. You can see that there exists a critical value of the gain p_{crit}. The system (5.40) is stable for gains less then critical one and unstable for gains greater then critical.

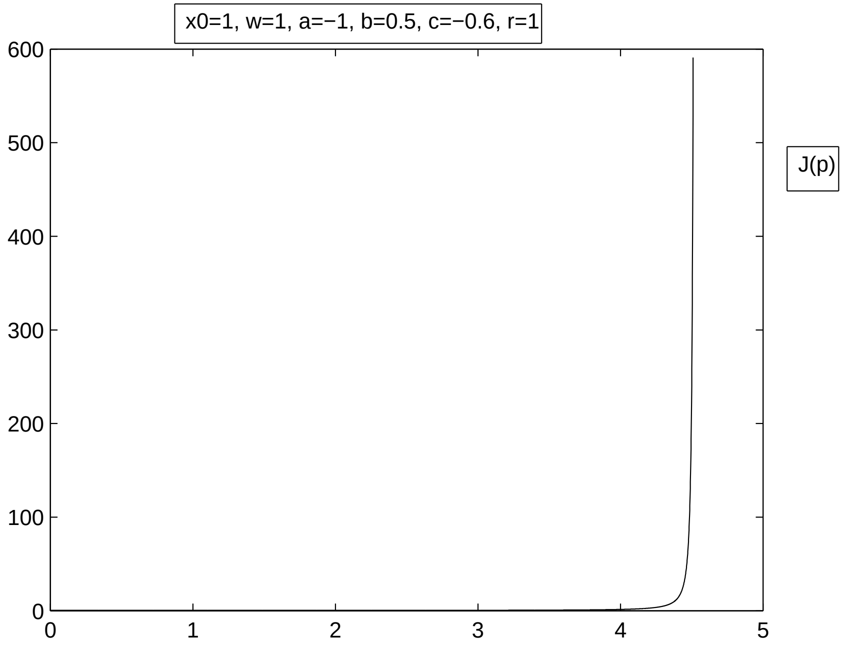

Fig. 5.1. Value of the index $J(p)$

Figure 5.2 shows the value of the index $J(p)$ for p less then critical gain. You can see that the function $J(p)$ is convex and has a minimum.

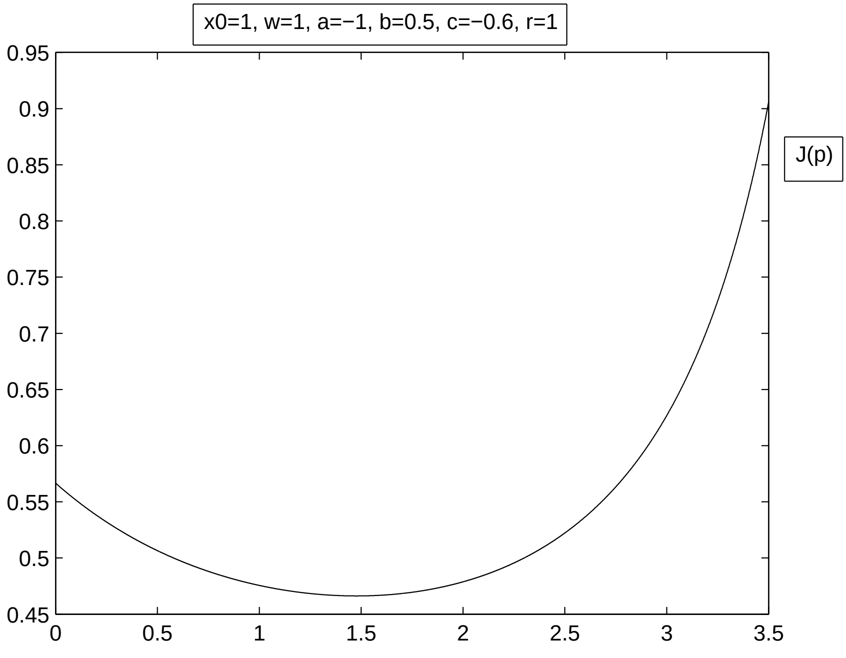

Fig. 5.2. Value of the index $J(p)$

We search for an optimal gain which minimize the index (5.53). Optimization results, obtained by means of the Matlab function *fminsearch*, are given in Table 5.1. These results are obtained for $x_0 = 1$, $w = 1$, $a = -1$, $b = 0.5$, and $c = -0.6$.

Table 5.1
Optimization results

Delay r	Optimal gain	Index value	Critical gain
1	2.2674	0.4598	4.51
2	1.4503	0.4959	2.95
3	1.2758	0.4995	2.50
4	1.2254	0.4999	2.32
5	1.2089	0.5000	2.20

5.1.5.2 Inertial system with delay and a PD-controller

Let us consider inertial system with delay and a PD-controller [23]

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{q}{T}x(t) + k_0 u(t-r) \\ u(t) = -px(t) - T_d \dfrac{dx(t)}{dt} \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.54}$$

$t \geq 0$, $x(t), u(t) \in \mathbb{R}$, $\theta \in [-r, 0]$, $r \geq 0$, p and T_d are parameters of a PD-controller, k_0 is a gain of a plant, T is a system time constant, φ is an initial function. In the case $q = 1$ equation (5.54) describes a static object and in the case $q = 0$ an astatic object.

One can reshape equation (5.54) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} + T_d k_0 \dfrac{dx(t-r)}{dt} = -\dfrac{q}{T}x(t) - k_0 p x(t-r) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.55}$$

for $t \geq 0$ and $\theta \in [-r, 0]$.

In parametric optimization problem we use the performance index

$$J = \int_0^\infty w x^2(t, \varphi) dt \tag{5.56}$$

where $w > 0$ and $x(t, \varphi)$ is a solution of (5.55) for initial function φ.

System of equations (4.20) takes a form

$$\begin{bmatrix} \dfrac{d}{d\xi}U(\xi) \\ \dfrac{d}{d\xi}Z(\xi) \end{bmatrix} = \begin{bmatrix} -\dfrac{\dfrac{q}{T} + k_0^2 T_d p}{1 - k_0^2 T_d^2} & -\dfrac{k_0 p + \dfrac{q k_0 T_d}{T}}{1 - k_0^2 T_d^2} \\ \dfrac{k_0 p + \dfrac{q k_0 T_d}{T}}{1 - k_0^2 T_d^2} & \dfrac{\dfrac{q}{T} + k_0^2 T_d p}{1 - k_0^2 T_d^2} \end{bmatrix} \begin{bmatrix} U(\xi) \\ Z(\xi) \end{bmatrix} \tag{5.57}$$

A fundamental matrix of solutions of equation (5.57) has a form

$$\Phi(\xi) = \begin{bmatrix} \cosh\lambda\xi - a_2 \sinh\lambda\xi & -a_1 \sinh\lambda\xi \\ a_1 \sinh\lambda\xi & \cosh\lambda\xi + a_2 \sinh\lambda\xi \end{bmatrix} \tag{5.58}$$

where

$$\lambda = \sqrt{\frac{\dfrac{q^2}{T^2} - k_0^2 p^2}{1 - k_0^2 T_d^2}},\ a_1 = \frac{k_0 p + \dfrac{q k_0 T_d}{T}}{\lambda(1 - k_0^2 T_d^2)},\ a_2 = \frac{\dfrac{q}{T} + k_0^2 T_d p}{\lambda(1 - k_0^2 T_d^2)} \tag{5.59}$$

Initial conditions of system (5.57) one obtains solving of the algebraic equation

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} U(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} -w \\ 0 \end{bmatrix} \tag{5.60}$$

where

$$q_{11} = -2\Big(\frac{q}{T} + k_0^2 T_d p\Big)$$
$$q_{12} = -2\Big(k_0 p + \frac{q k_0 T_d}{T}\Big)$$
$$q_{21} = \frac{k_0 p + \frac{q k_0 T_d}{T}}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r - 1$$
$$q_{22} = \cosh \lambda r + \frac{\frac{q}{T} + k_0^2 T_d p}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r$$

Solving equation (5.60) we obtain

$$U(0) = \frac{w}{M}\left[\cosh \lambda r + \frac{\frac{q}{T} + k_0^2 T_d p}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r\right] \tag{5.61}$$

$$Z(0) = \frac{w}{M}\left[1 - \frac{k_0 p + \frac{q k_0 T_d}{T}}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r\right] \tag{5.62}$$

where

$$M = 2\left[\Big(\frac{q}{T} + k_0^2 T_d p\Big)\cosh \lambda r + \lambda(1 - k_0^2 T_d^2)\sinh \lambda r + k_0 p + \frac{q k_0 T_d}{T}\right] \tag{5.63}$$

Solution of equation (5.57) has a form

$$U(\xi) = \frac{w}{M}\left[\cosh \lambda r + \frac{\frac{q}{T} + k_0^2 T_d p}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r\right] \cosh \lambda \xi - \frac{w}{2\lambda(1 - k_0^2 T_d^2)} \sinh \lambda \xi \tag{5.64}$$

$$Z(\xi) = \frac{w}{M}\left[1 - \frac{k_0 p + \frac{q k_0 T_d}{T}}{\lambda(1 - k_0^2 T_d^2)} \sinh \lambda r\right] \cosh \lambda \xi +$$
$$+ \frac{w}{M\lambda(1 - k_0^2 T_d^2)}\left[\frac{q}{T} + k_0^2 T_d p + \Big(k_0 p + \frac{q k_0 T_d}{T}\Big)\cosh \lambda r\right] \sinh \lambda \xi \tag{5.65}$$

We compute the value of the performance index (5.56) for initial function φ given by a term

$$\varphi(\theta) = \begin{cases} x_0 & \text{for } \theta = 0 \\ 0 & \text{for } \theta \in [-r, 0) \end{cases} \tag{5.66}$$

The value of functional (5.11) for φ given by formula (5.66) is equal to

$$J = x_0^2 \left[(1 + k_0^2 T_d^2) U(0) + 2k_0 T_d Z(0) \right] \tag{5.67}$$

After calculations one obtains

$$J = \frac{\dfrac{wx_0^2}{2}\left(2k_0 T_d + (1 + k_0^2 T_d^2) \cosh \lambda r + \dfrac{\dfrac{q}{T} - k_0^2 T_d p}{\lambda} \sinh \lambda r \right)}{k_0 p + \dfrac{q k_0 T_d}{T} + \left(\dfrac{q}{T} + k_0^2 T_d p \right) \cosh \lambda r + \lambda (1 - k_0^2 T_d^2) \sinh \lambda r} \tag{5.68}$$

Figure 5.3 shows the value of the index $J(p)$ for fixed $T_d = 0.4733$ and $r = 1$. You can see that there exists a critical value of the gain p_{crit}. The system (5.55) is stable for gains less then critical one and unstable for gains grater then critical.

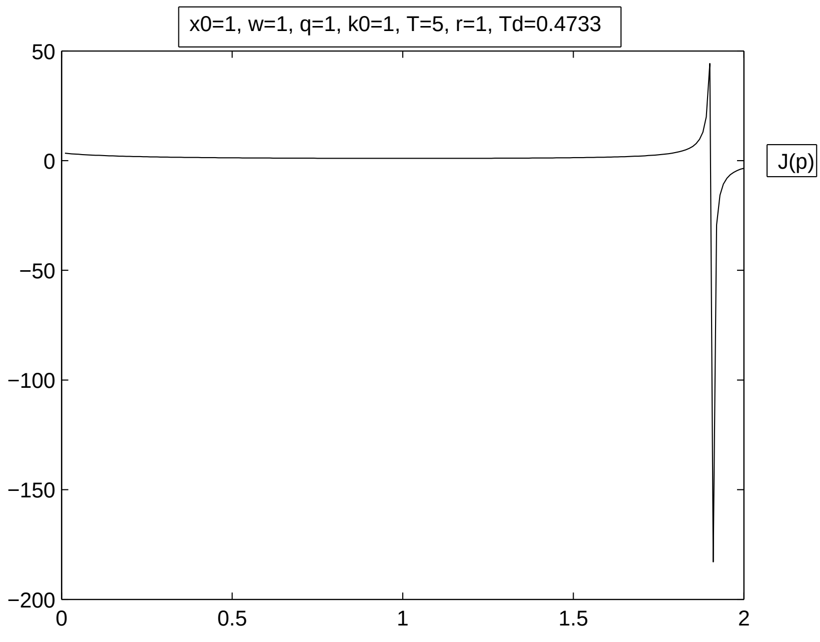

Fig. 5.3. Value of the index $J(p)$ for fixed $T_d = 0.4733$ and $r = 1$

Figure 5.4 shows the value of the index $J(p)$ for fixed $T_d = 0.4733$, $r = 1$ and gains less the critical one. You can see that the function $J(p)$ is convex and has a minimum.

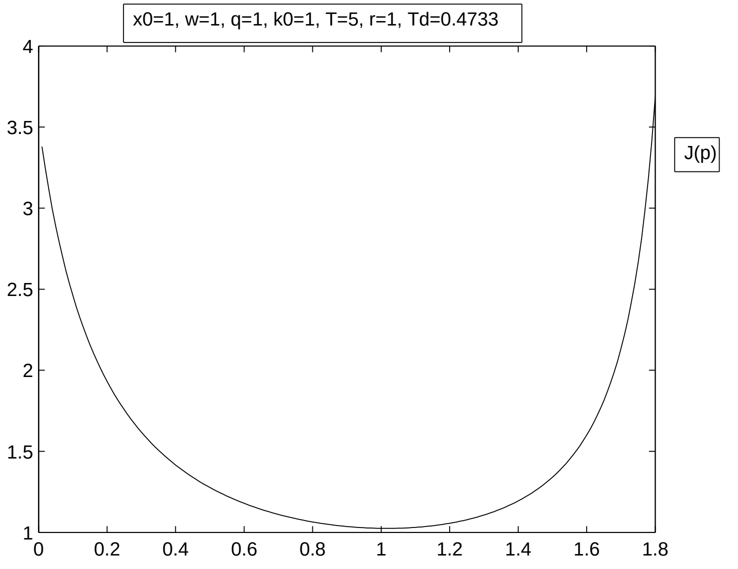

Fig. 5.4. Value of the index $J(p)$ for fixed $T_d = 0.4733$ and $r = 1$

Figure 5.5 shows the value of the index $J(T_d)$ for fixed $p = 1.0168$ and $r = 1$. There exists a critical value of the differential time $T_{d\,crit}$ too, which determines the interval of stability.

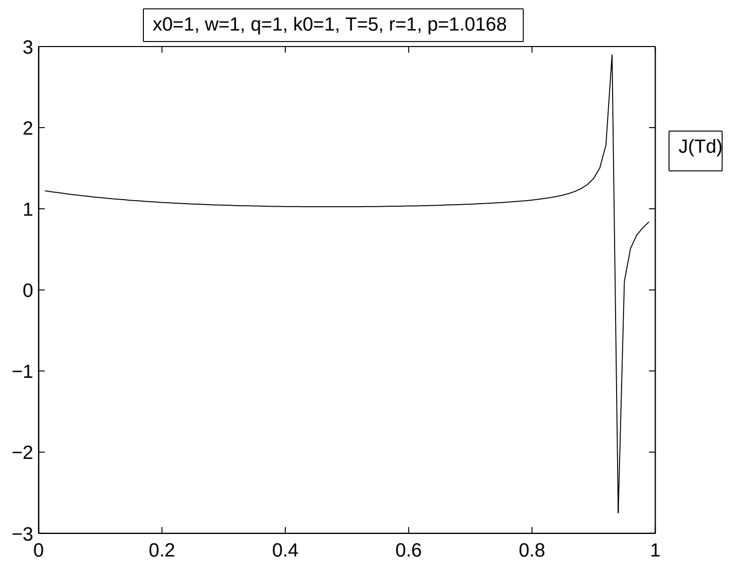

Fig. 5.5. Value of the index $J(T_d)$ for fixed $p = 1.0168$ and $r = 1$

Figure 5.6 shows the value of the index $J(T_d)$ for fixed $p = 1.0168$, $r = 1$ and T_d less the critical one. You can see that the function $J(T_d)$ is convex and has a minimum.

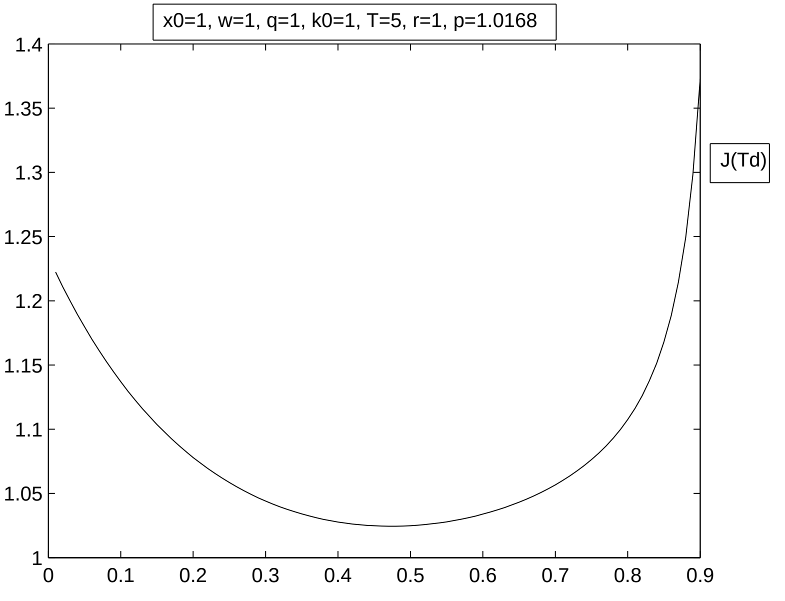

Fig. 5.6. Value of the index $J(T_d)$ for fixed $p = 1.0168$ and $r = 1$

We search for an optimal parameters of a PD-controller which minimize the index (5.68). Optimization results, obtained by means of Matlab function *fminsearch*, are given in Table 5.2. These results are obtained for $x_0 = 1$, $w = 1$, $q = 1$, $T = 5$, and $k_0 = 1$.

Table 5.2
Optimization results

Delay r	Optimal p	Optimal T_d	Index value
1.0	1.0168	0.4733	1.0245
1.5	0.6687	0.4559	1.3567
2.0	0.4949	0.4389	1.6096
2.5	0.3907	0.4222	1.8035
3.0	0.3211	0.4058	1.9528
3.5	0.2714	0.3897	2.0685
4.0	0.2340	0.3739	2.1586

Critical values p_{crit} and $T_{d\,crit}$ depend on the value of time delay. This dependence is presented in Table 5.3. Critical gain is obtained for fixed $T_d = 0.4733$ and critical differential time is obtained for fixed $p = 0.45$.

Table 5.3
Critical gain and differential time

Delay r	p_{crit}	$T_{d\,crit}$
1.0	1.86	0.98
1.5	1.25	0.97
2.0	0.95	0.95
2.5	0.77	0.92
3.0	0.65	0.87
3.5	0.56	0.81
4.0	0.50	0.71

5.2 Neutral system with two delays

5.2.1 Mathematical model of neutral system with two delays

Let us consider a neutral system with two delays

$$\begin{cases} \dfrac{dx(t)}{dt} - D\dfrac{dx(t-h)}{dt} = Ax(t) + Bx(t-h) + Cx(t-r) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.69}$$

for $t \geq 0$, $\theta \in [-r, 0]$.
The state $x(t) \in \mathbb{R}^n$, matrices A, B, C, $D \in \mathbb{R}^{n\times n}$, initial function $\varphi \in PC^1([-r,0],\mathbb{R}^n)$ – the space of piece-wise continuous vector valued functions defined on the segment $[-r, 0]$ with the uniform norm $\| \varphi \|_{PC^1} = \sup\limits_{\theta\in[-r,0]} \| \varphi(\theta) \|$, delays $r > h > 0$.
We assume that the difference $x(t) - Dx(t-h)$ is continuous and differentiable for $t \geq 0$, except possibly a countable number of points, $t_j = jh$, $j = 0, 1, 2, ...$

Definition 5.3. *The* ***difference equation*** *associated with (5.69) is given by a term*

$$x(t) - Dx(t-h) = 0 \tag{5.70}$$

for $t \geq 0$.

Let $x(t, \varphi)$ be the solution of system (5.69) with the initial function φ for $t \geq 0$.

Definition 5.4. [2] $K(t)$ *is the* ***fundamental matrix*** *of system (5.69) if it satisfies the matrix equation*

$$\frac{d}{dt}K(t) - D\frac{d}{dt}K(t-h) = AK(t) + BK(t-h) + CK(t-r) \tag{5.71}$$

for $t \geq 0$ *and the following conditions*

- *initial condition*: $K(0) = I_{n\times n}$ and $K(t) = 0_{n\times n}$ for $t < 0$ where $I_{n\times n}$ is the identity $n\times n$ matrix and $0_{n\times n}$ is the zero $n\times n$ matrix,
- *sewing condition*: $K(t) - DK(t-h)$ is continuous for $t > 0$.

Theorem 5.3. [2] *Let* $K(t)$ *be the fundamental matrix of system (5.69), then for* $t \geq 0$

$$x(t,\varphi) = [K(t) - K(t-h)D]\,\varphi(0) + \int_{-h}^{0} K(t-h-\theta)\left[B\varphi(\theta) + D\frac{d}{d\theta}\varphi(\theta)\right]d\theta +$$

$$+\int_{-r}^{0} K(t-r-\theta)C\varphi(\theta)d\theta \tag{5.72}$$

This expression is called *the Cauchy formula* for system (5.69).
It follows from the definition that the fundamental matrix $K(t)$ has discontinuity points.
The *sewing condition* can be written in a form

$$K(t+0) - DK(t+0-h) = K(t-0) - DK(t-0-h) \tag{5.73}$$

for $t > 0$.
Formula (5.73) gives *the jump equation*

$$\triangle K(t) - D\triangle K(t-h) = 0 \tag{5.74}$$

for $t \geq 0$, where $\triangle K(t) = K(t+0) - K(t-0)$

Theorem 5.4. *The fundamental matrix* $K(t)$ *has jumps at points* $t_j = jh,\ j = 0,1,2,...$

$$\triangle K(t)\,|_{t=t_j} = K(jh+0) - K(jh-0) = D^j \tag{5.75}$$

and $K(t) = K(t+0)$ *at the jump points.*

Proof. We solve the jump equation (5.74) at $t_j = jh,\ j = 0,1,2,..$, with the initial condition $\triangle K(0)$

$$\Delta K(0) = K(0+0) - K(0-0) = I - 0 = I$$

$$\Delta K(h) = D\Delta K(0) = D$$

$$\Delta K(2h) = D\Delta K(h) = D^2$$

$$\Delta K(jh) = D\Delta K(jh-h) = D^j$$

□

Theorem 5.5. [2] *The fundamental matrix $K(t)$ of system (5.69) satisfies also the equation*

$$\frac{d}{dt}K(t) - \frac{d}{dt}K(t-h)D = K(t)A + K(t-h)B + K(t-r)C \tag{5.76}$$

for $t > 0$ and $t \neq jh,\ j = 1,2,...$

The initial condition for equation (5.69) can be written in a form

$$x_t(\varphi)\mid_{t=0} = \varphi \tag{5.77}$$

where $x_t \in PC^1([-r,0],\mathbb{R}^n)$ is a shifted restriction of the function $x(\cdot,\varphi)$ to the segment $[-r,0]$. The eigenvalues of neutral equation (5.69) for large modulus are asymptotically equal to the eigenvalues of the difference equation (5.70).

According to the Theorem 9.6.1 [40] the difference equation (5.70) is stable when the matrix D is Schur stable. When the matrix D is Schur stable, then the asymptotic stability of system (5.69) is equivalent to the exponential stability of system (5.69). We assume that D is not singular and a Schur stable matrix.

5.2.2 The Lyapunov–Krasovskii functional for a neutral system with two delays

Problem 5.2. *Given a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$. We are looking for a functional $v : PC^1([-r,0],\mathbb{R}^n) \to \mathbb{R}$ such that along the solutions of system (5.69) the following equality holds*

$$\frac{d}{dt}v(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi) \tag{5.78}$$

for $t \geq 0$, where $x(t,\varphi)$ is a solution of system (5.69), with the initial function $\varphi \in PC^1([-r,0],\mathbb{R}^n)$, given by (5.72) and $x_t(\varphi)$ is a shifted restriction of $x(\cdot,\varphi)$ to an interval $[t-r,t]$.

We assume that system (5.69) is exponentially stable and integrate both sides of equation (5.78) from 0 to infinity. We obtain

$$v(\varphi) = v(x_t(\varphi)\mid_{t=0}) = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)dt \tag{5.79}$$

Taking into account (5.72) we calculate the integral of the right-hand side of equation (5.79)

$$\int_0^\infty x^T(t,\varphi)Wx(t,\varphi)dt = \varphi^T(0)\int_0^\infty K^T(t)WK(t)dt\varphi(0) - \varphi^T(0)\int_0^\infty K^T(t)WK(t-h)dtD\varphi(0)+$$

$$-\varphi^T(0)D^T\int_0^\infty K^T(t-h)WK(t)dt\varphi(0) + \varphi^T(0)D^T\int_0^\infty K^T(t-h)WK(t-h)dtD\varphi(0)+$$

$$+2\varphi^T(0)\int_{-h}^0\left[\int_0^\infty K^T(t)WK(t-h-\theta)dt\right]\left[B\varphi(\theta) + D\frac{d}{d\theta}\varphi(\theta)\right]d\theta+$$

$$-2\varphi^T(0)D^T\int_{-h}^{0}\left[\int_0^\infty K^T(t-h)WK(t-h-\theta)dt\right]\left[B\varphi(\theta)+D\frac{d}{d\theta}\varphi(\theta)\right]d\theta+$$

$$+2\varphi^T(0)\int_{-r}^{0}\left[\int_0^\infty K^T(t)WK(t-r-\theta)dt\right]C\varphi(\theta)d\theta+$$

$$-2\varphi^T(0)D^T\int_{-r}^{0}\left[\int_0^\infty K^T(t-h)WK(t-r-\theta)dt\right]C\varphi(\theta)d\theta+$$

$$+\int_{-h}^{0}\int_{-h}^{0}\left[\varphi^T(\theta)B^T+\frac{d}{d\theta}\varphi^T(\theta)D^T\right]\left[\int_0^\infty K^T(t-h-\theta)WK(t-h-\xi)dt\right]\times$$

$$\times\left[B\varphi(\xi)D\frac{d}{d\xi}\varphi(\xi)\right]d\theta d\xi+$$

$$+2\int_{-h}^{0}\int_{-r}^{0}\left[\varphi^T(\theta)B^T+\frac{d}{d\theta}\varphi^T(\theta)D^T\right]\left[\int_0^\infty K^T(t-h-\theta)WK(t-r-\xi)dt\right]C\varphi(\xi)d\theta d\xi+$$

$$+\int_{-r}^{0}\int_{-r}^{0}\varphi^T(\theta)C^T\left[\int_0^\infty K^T(t-r-\theta)WK(t-r-\xi)dt\right]C\varphi(\xi)d\theta d\xi \tag{5.80}$$

Using a Lyapunov matrix $U(\xi)$ (4.8) we attain a formula for the functional $v(\varphi)$

$$v(\varphi)=\int_0^\infty x^T(t,\varphi)Wx(t,\varphi)dt=$$

$$=\varphi^T(0)\left[U(0)-U(-h)D-D^TU^T(-h)+D^TU(0)D\right]\varphi(0)+$$

$$+2\varphi^T(0)\int_{-h}^{0}\left[U(-\theta-h)-D^TU(-\theta)\right]\left[B\varphi(\theta)+D\frac{d}{d\theta}\varphi(\theta)\right]d\theta+$$

$$+2\varphi^T(0)\int_{-r}^{0}\left[U(-\theta-r)-D^TU(h-r-\theta)\right]C\varphi(\theta)d\theta+$$

$$+\int_{-h}^{0}\int_{-h}^{0}\left[B\varphi(\theta)+D\frac{d}{d\theta}\varphi(\theta)\right]^TU(\theta-\xi)\left[B\varphi(\xi)+D\frac{d}{d\xi}\varphi(\xi)\right]d\theta d\xi+$$

$$+2\int_{-h}^{0}\int_{-r}^{0}\left[B\varphi(\theta)+D\frac{d}{d\theta}\varphi(\theta)\right]^TU(\theta-\xi+h-r)C\varphi(\xi)d\theta d\xi+$$

$$+\int_{-r}^{0}\int_{-r}^{0}\varphi^T(\theta)C^TU(\theta-\xi)C\varphi(\xi)d\theta d\xi \tag{5.81}$$

Theorem 5.6. *Let system (5.69) be exponentially stable, $U(\xi)$ be a Lyapunov matrix associated with a symmetric matrix W. The functional (5.81) solves Problem 40.*

5.2.3 Formulation of the parametric optimization problem for a neutral system with two delays

Let us consider a neutral system with two delays and a P-controller

$$\begin{cases} \frac{dx(t)}{dt} - D\frac{dx(t-h)}{dt} = Ax(t) + Bx(t-h) + C_1 u(t-r) \\ u(t) = -Px(t) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.82}$$

for $t \geq 0,\ \theta \in [-r,0]$
Where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $A,B,D \in \mathbb{R}^{n\times n}$, $C_1 \in \mathbb{R}^{n\times p}$, $P \in \mathbb{R}^{p\times n}$ is a P-controller gain, $\varphi \in PC^1([-r,0],\mathbb{R}^n)$
System (5.82) can be written in the equivalent form

$$\begin{cases} \frac{dx(t)}{dt} - D\frac{dx(t-h)}{dt} = Ax(t) + Bx(t-h) - C_1 Px(t-r) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.83}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)dt \tag{5.84}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,\varphi)$ is a solution of equation (5.83) for initial function φ.

Problem 5.3. *Determine the matrix $P \in \mathbb{R}^{p\times n}$ whose minimize the integral quadratic performance index of quality (5.84).*

The value of the performance index of quality (5.84) is equal to the value of the functional (5.81) for initial function φ, in which a matrix C should be replaced by a matrix $-C_1P$. To calculate the value of the functional (5.81) we need a Lyapunov matrix $U(\xi)$.

5.2.4 The Lyapunov matrix for a neutral system with two delays

Let system (5.69) be exponentially stable. The Lyapunov matrix $U(\xi)$ is given by equation (4.8). We will compute the derivative of $U(\xi)$ with respect to ξ. According to Theorem 5.4 the matrix $K(t)$ has jumps at points $t_j = jh,\ j = 0,1,2,...$ We take a positive value $\xi \neq jh$ for $j = 0,1,2,...$ It can be written in a form $\xi = lh + \eta$, where $\eta \in (0,h)$, and $l = 1,2,...$

For $t \geq 0$ the matrix $K(t+\xi)$ has jumps at points $t_j = jh - \eta$ for $j = 1,2,...$ On the set $G = [0,\infty) \setminus \{t_j\}_{j=1}^{\infty}$ the matrix $K(t+\xi)$ has no jumps.
We can compute the derivative

$$\frac{d}{d\xi}U(\xi)\,|_{\xi=lh+\eta} = \int_G K^T(t)W\frac{\partial}{\partial\xi}K(t+\xi)dt +$$
$$+\sum_{j=1}^{\infty} K^T(t_j)W[K(t_j+\xi+0) - K(t_j+\xi-0)] \tag{5.85}$$

Using equation (5.75) from Theorem 38 we calculate

$$K(t_j+\xi+0) - K(t_j+\xi-0) = K((j+l)h+0) - K((j+l)h-0) = D^{l+j} \tag{5.86}$$

Taking into account equation (5.86) we obtain

$$\frac{d}{d\xi}U(\xi) = \int_G K^T(t)W\frac{\partial}{\partial\xi}K(t+\xi)dt + \sum_{j=1}^{\infty} K^T(t_j)WD^{l+j} \tag{5.87}$$

$$\frac{d}{d\xi}U(\xi-h) = \int_G K^T(t)W\frac{\partial}{\partial\xi}K(t+\xi-h)dt + \sum_{j=1}^{\infty} K^T(t_j)WD^{l-1+j} \tag{5.88}$$

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi-h)D = \int_G K^T(t)W\Big[\frac{\partial}{\partial\xi}K(t+\xi) - \frac{\partial}{\partial\xi}K(t+\xi-h)D\Big]dt \tag{5.89}$$

Equation (5.76) for $t+\xi$ takes a form

$$\frac{d}{dt}K(t+\xi) - \frac{d}{dt}K(t+\xi-h)D = K(t+\xi)A + K(t+\xi-h)B + K(t+\xi-r)C \tag{5.90}$$

We substitute the right-hand side of (5.90) into equation (5.89) and obtain

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi-h)D = \int_G K^T(t)WK(t+\xi)dtA + \int_G K^T(t)WK(t+\xi-h)dtB +$$
$$+\int_G K^T(t)WK(t+\xi-r)dtC = U(\xi)A + U(\xi-h)B + U(\xi-r)C \tag{5.91}$$

We have obtained *the dynamic property* of Lyapunov matrix

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi-h)D = U(\xi)A + U(\xi-h)B + U(\xi-r)C \tag{5.92}$$

for $\xi \geq 0$ and $\xi \neq jh,\ j = 0,1,2,...$

Now we introduce *the algebraic property* of Lyapunov matrix

$$\frac{d}{dt}\Big[K(t)-K(t-h)D\Big]^T W[K(t)-K(t-h)D] =$$
$$= \Big[\frac{d}{dt}K(t)-\frac{d}{dt}K(t-h)D\Big]^T W[K(t)-K(t-h)D]+$$
$$+[K(t)-K(t-h)D]^T W\Big[\frac{d}{dt}K(t)-\frac{d}{dt}K(t-h)D\Big] \tag{5.93}$$

We substitute the right-hand side of equation (5.76) into equation (5.93) and integrate both sides with respect to t from zero to infinity taking into account the definition of Lyapunov matrix (4.8). After calculation we obtain *the algebraic property* of Lyapunov matrix

$$-W = A^T U(0)+U(0)A-A^T U(-h)D-D^T U^T(-h)A+B^T U^T(-h)+U(-h)B+$$
$$-B^T U(0)D-D^T U(0)B+C^T U^T(-r)+U(-r)C-C^T U(r-h)D-D^T U(h-r)C \tag{5.94}$$

We calculate $U(-\xi)$

$$U(-\xi) = \int_0^\infty K^T(t)WK(t-\xi)dt = \int_{-\xi}^\infty K^T(\eta+\xi)WK(\eta)d\eta = \int_{-\xi}^0 K^T(\eta+\xi)WK(\eta)d\eta+$$
$$+\int_0^\infty K^T(\eta+\xi)WK(\eta)d\eta = \int_0^\infty K^T(\eta+\xi)WK(\eta)d\eta =$$
$$= \Big[\int_0^\infty K^T(\eta)WK(\eta+\xi)d\eta\Big]^T = U^T(\xi) \tag{5.95}$$

The integral $\int_{-\xi}^0 K^T(\eta+\xi)WK(\eta)d\eta = 0$ because $K(\eta)=0$ for $\eta<0$.
We have obtained *the symmetry property* of Lyapunov matrix

$$U(-\xi) = U^T(\xi) \tag{5.96}$$

for $\xi \geq 0$.
We had obtained the following theorem.

Theorem 5.7. *Let system (5.69) be exponentially stable.The Lyapunov matrix for that system fulfills the conditions:*

$$\frac{d}{d\xi}U(\xi)-\frac{d}{d\xi}U(\xi-h)D = U(\xi)A+U(\xi-h)B+U(\xi-r)C \tag{5.97}$$

for $\xi \geq 0$ *and* $\xi \neq jh,\ j=0,1,2,...$

$$U(-\xi) = U^T(\xi) \tag{5.98}$$

for $\xi \geq 0$

$$-W = A^T U(0)+U(0)A-A^T U(-h)D-D^T U^T(-h)A+B^T U^T(-h)+U(-h)B+$$
$$-B^T U(0)D-D^T U(0)B+C^T U^T(-r)+U(-r)C-C^T U(r-h)D-D^T U(h-r)C \tag{5.99}$$

5.2.5 The Lyapunov matrix for a neutral system with two commensurate delays

Let us consider a neutral system with two commensurate delays

$$\begin{cases} \dfrac{dx(t)}{dt} - D\dfrac{dx(t-h)}{dt} = Ax(t) + Bx(t-h) + Cx(t-2h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.100}$$

for $t \geq 0,\ \theta \in [-2h, 0]$.
The state $x(t) \in \mathbb{R}^n$, matrices A, B, C, $D \in \mathbb{R}^{n\times n}$, initial function $\varphi \in PC^1([-2h,0],\mathbb{R}^n)$ – the space of piece-wise continuous vector valued functions defined on the segment $[-2h,0]$ with the uniform norm $\| \varphi \|_{PC^1} = \sup\limits_{\theta\in[-2h,0]} \| \varphi(\theta) \|$, delays h, $2h > 0$.
The set of equations (5.97), (5.98), (5.99) for system (5.100) takes a form

$$\frac{d}{d\tau}U(\tau) - \frac{d}{d\tau}U(\tau-h)D = U(\tau)A + U(\tau-h)B + U(\tau-2h)C \tag{5.101}$$

$$U(-\tau) = U^T(\tau) \tag{5.102}$$

for $\tau \in [0,2h]$

$$\begin{aligned} -W = {} & A^TU(0) + U(0)A - A^TU(-h)D - D^TU^T(-h)A + B^TU^T(-h) + U(-h)B + \\ & -B^TU(0)D - D^TU(0)B + C^TU^T(-2h) + U(-2h)C - C^TU(h)D - D^TU(-h)C \end{aligned} \tag{5.103}$$

Formula (5.102) extends the function U defined on the segment $[0,2h]$ to the segment $[-2h,0]$. Indeed for $\tau \in [0,2h]$, $U(-\tau) = U^T(\tau)$. For $\varsigma = -\tau$, $U(\varsigma) = U^T(-\varsigma)$ and $\varsigma \in [-2h,0]$.
We define the functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$ for $\xi \in [0,h]$

$$U_1(\xi) = U(\xi) \tag{5.104}$$

$$U_2(\xi) = U(h+\xi) \tag{5.105}$$

$$Z_1(\xi) = U(\xi-h) = U^T(-\xi+h) \tag{5.106}$$

$$Z_2(\xi) = U(\xi-2h) = U^T(-\xi+2h) \tag{5.107}$$

Relations (5.104)–(5.107) imply

$$U(0) = U_1(0), \quad U(-h) = Z_1(0)$$

$$U(-2h) = Z_2(0), \quad U(h) = U_2(0) \tag{5.108}$$

Taking into account (5.108) the algebraic property (5.103) can be written in a form

$$-W = A^T U_1(0) + U_1(0)A - A^T Z_1(0)D - D^T Z_1^T(0)A + B^T Z_1^T(0) + Z_1(0)B +$$
$$-B^T U_1(0)D - D^T U_1(0)B + C^T Z_2^T(0) + Z_2(0)C - C^T U_2(0)D - D^T Z_1(0)C \qquad (5.109)$$

We will use the relations

$$U(-\xi) = U^T(\xi) = U_1^T(\xi) \qquad (5.110)$$

$$U(-\xi - h) = U^T(\xi + h) = U_2^T(\xi) \qquad (5.111)$$

$$U(2h - \xi) = U^T(\xi - 2h) = Z_2^T(\xi) \qquad (5.112)$$

for $\xi \in [0,h]$.
Taking into account relations (5.104)–(5.107), equation (5.101) for $\tau = \xi$, $d\tau = d\xi$, $\xi \in [0,h]$ can be written in a form

$$\frac{d}{d\xi}U_1(\xi) - \frac{d}{d\xi}Z_1(\xi)D = U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C \qquad (5.113)$$

Taking into account relations (5.104)–(5.107), equation (5.101) for $\tau = \xi + h$, $d\tau = d\xi$, $\xi \in [0,h]$ can be written in a form

$$\frac{d}{d\xi}U_2(\xi) - \frac{d}{d\xi}U_1(\xi)D = U_2(\xi)A + U_1(\xi)B + Z_1(\xi)C \qquad (5.114)$$

Equation (5.101) for $\tau = -\xi + h$, $d\tau = -d\xi$, $\xi \in [0,h]$ can be written in a form

$$\frac{d}{d\xi}U(-\xi + h) - \frac{d}{d\xi}U(-\xi)D = -U(-\xi + h)A - U(-\xi)B - U(-\xi - h)C \qquad (5.115)$$

We transpose doth sides of equation (5.115) and taking into account relations (5.104)–(5.107), (5.110) and (5.111) we obtain

$$\frac{d}{d\xi}Z_1(\xi) - D^T\frac{d}{d\xi}U_1(\xi) = -A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi) \qquad (5.116)$$

Equation (5.101) for $\tau = -\xi + 2h$, $d\tau = -d\xi$, $\xi \in [0,h]$ can be written in a form

$$\frac{d}{d\xi}U(-\xi + 2h) - \frac{d}{d\xi}U(-\xi + h)D = -U(-\xi + 2h)A - U(-\xi + h)B - U(-\xi)C \quad (5.117)$$

We transpose both sides of equation (5.117) and taking into account relations (5.104)–(5.107), (5.110) and (5.112) we obtain

$$\frac{d}{d\xi}Z_2(\xi) - D^T\frac{d}{d\xi}Z_1(\xi) = -A^T Z_2(\xi) - B^T Z_1(\xi) - C^T U_1(\xi) \qquad (5.118)$$

Equations (5.113) and (5.116) can be reshape to a form

$$\frac{d}{d\xi}U_1(\xi) - D^T\frac{d}{d\xi}U_1(\xi)D = -A^T Z_1(\xi)D - B^T U_1(\xi)D + \\ -C^T U_2(\xi)D + U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C \tag{5.119}$$

$$\frac{d}{d\xi}Z_1(\xi) - D^T\frac{d}{d\xi}Z_1(\xi)D = D^T U_1(\xi)A + D^T Z_1(\xi)B + \\ +D^T Z_2(\xi)C - A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi) \tag{5.120}$$

We have obtained the set of ordinary differential equations with unknown $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$.

$$\begin{cases} \frac{d}{d\xi}U_1(\xi) - D^T\frac{d}{d\xi}U_1(\xi)D = -A^T Z_1(\xi)D - B^T U_1(\xi)D + \\ -C^T U_2(\xi)D + U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C \\ \frac{d}{d\xi}U_2(\xi) - \frac{d}{d\xi}U_1(\xi)D = U_2(\xi)A + U_1(\xi)B + Z_1(\xi)C \\ \frac{d}{d\xi}Z_1(\xi) - D^T\frac{d}{d\xi}Z_1(\xi)D = D^T U_1(\xi)A + D^T Z_1(\xi)B + \\ +D^T Z_2(\xi)C - A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi) \\ \frac{d}{d\xi}Z_2(\xi) - D^T\frac{d}{d\xi}Z_1(\xi) = -A^T Z_2(\xi) - B^T Z_1(\xi) - C^T U_1(\xi) \end{cases} \tag{5.121}$$

for $\xi \in [0,h]$ with initial conditions

$$U_1(0), U_2(0), Z_1(0), Z_2(0)$$

Equation (5.121) can be written in a form

$$\frac{d}{d\xi}\begin{bmatrix} colU_1(\xi) \\ colU_2(\xi) \\ colZ_1(\xi) \\ colZ_2(\xi) \end{bmatrix} = \mathscr{H}\begin{bmatrix} colU_1(\xi) \\ colU_2(\xi) \\ colZ_1(\xi) \\ colZ_2(\xi) \end{bmatrix} \tag{5.122}$$

for $\xi \in [0,h]$ with initial conditions

$$colU_1(0), colU_2(0), colZ_1(0), colZ_2(0)$$

Solution of the set of ordinary differential equations (5.122) is given in a form

$$\begin{bmatrix} colU_1(\xi) \\ colU_2(\xi) \\ colZ_1(\xi) \\ colZ_2(\xi) \end{bmatrix} = \Phi(\xi)\begin{bmatrix} colU_1(0) \\ colU_2(0) \\ colZ_1(0) \\ colZ_2(0) \end{bmatrix} \tag{5.123}$$

where a matrix

$$\Phi(\xi)=\begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) & \Phi_{23}(\xi) & \Phi_{24}(\xi) \\ \Phi_{31}(\xi) & \Phi_{32}(\xi) & \Phi_{33}(\xi) & \Phi_{34}(\xi) \\ \Phi_{41}(\xi) & \Phi_{42}(\xi) & \Phi_{43}(\xi) & \Phi_{44}(\xi) \end{bmatrix} \tag{5.124}$$

is a fundamental matrix of system (5.122).
We determine the initial conditions $colU_1(0)$, $colU_2(0)$, $colZ_1(0)$, $colZ_2(0)$.
Relations (5.104)–(5.107) imply

$$U_1(h)=U(h)=U_2(0) \tag{5.125}$$

$$Z_1(h)=U(0)=U_1(0) \tag{5.126}$$

$$Z_2(h)=U(-h)=Z_1(0) \tag{5.127}$$

Solution of the differential equations (5.122) for $\xi=h$ is given

$$\begin{aligned} colU_1(h)=colU_2(0)=\Phi_{11}(h)colU_1(0)+\Phi_{12}(h)colU_2(0)+\\ +\Phi_{13}(h)colZ_1(0)+\Phi_{14}(h)colZ_2(0) \end{aligned} \tag{5.128}$$

$$\begin{aligned} colZ_1(h)=colU_1(0)=\Phi_{31}(h)colU_1(0)+\Phi_{32}(h)colU_2(0)+\\ +\Phi_{33}(h)colZ_1(0)+\Phi_{34}(h)colZ_2(0) \end{aligned} \tag{5.129}$$

$$\begin{aligned} colZ_2(h)=colZ_1(0)=\Phi_{41}(h)colU_1(0)+\Phi_{42}(h)colU_2(0)+\\ +\Phi_{43}(h)colZ_1(0)+\Phi_{44}(h)colZ_2(0) \end{aligned} \tag{5.130}$$

Equations (5.128) to (5.130) and (5.109) enables us to calculate the initial conditions of system (5.122). We reshape them to a form

$$\Phi_{11}(h)colU_1(0)+(\Phi_{12}(h)-1)colU_2(0)+\Phi_{13}(h)colZ_1(0)+\Phi_{14}(h)colZ_2(0)=0 \tag{5.131}$$

$$(\Phi_{31}(h)-1)colU_1(0)+\Phi_{32}(h)colU_2(0)+\Phi_{33}(h)colZ_1(0)+\Phi_{34}(h)colZ_2(0)=0 \tag{5.132}$$

$$\Phi_{41}(h)colU_1(0)+\Phi_{42}(h)colU_2(0)+(\Phi_{43}(h)-1)colZ_1(0)+\Phi_{44}(h)colZ_2(0)=0 \tag{5.133}$$

$$\begin{aligned} A^TU_1(0)+U_1(0)A-A^TZ_1(0)D-D^TZ_1^T(0)A+B^TZ_1^T(0)+Z_1(0)B-B^TU_1(0)D+\\ -D^TU_1(0)B+C^TZ_2^T(0)+Z_2(0)C-C^TU_2(0)D-D^TZ_1(0)C=-W \end{aligned} \tag{5.134}$$

5.2.6 The example

Let us consider a neutral system with a P-controller

$$\begin{cases} \dfrac{dx(t)}{dt} - d\dfrac{dx(t-h)}{dt} = ax(t) + bx(t-h) + c_1 u(t-2h) \\ u(t) = -px(t) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.135}$$

$t \geq 0$, $x(t), u(t) \in \mathbb{R}$, $\theta \in [-2h, 0]$, $h \geq 0$. The parameter p is a gain of a P-controller, φ is an initial function of system.
One can reshape equation (5.135) to a form

$$\begin{cases} \dfrac{dx(t)}{dt} - d\dfrac{dx(t-h)}{dt} = ax(t) + bx(t-h) - c_1 px(t-2h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{5.136}$$

for $t \geq 0$ and $\theta \in [-2h, 0]$.
In parametric optimization problem we use the performance index of quality

$$J = \int_0^\infty wx^2(t,\varphi)dt \tag{5.137}$$

where $w > 0$ and $x(t,\varphi)$ is a solution of equation (5.136) for initial function φ.
The Lyapunov functional for system (5.136) has a form, see formula (5.81)

$$\begin{gathered} v(\varphi) = [(1+d^2)U(0) - 2dU(-h)]\varphi^2(0) + \\ +2\varphi(0)\int_{-h}^{0}\left[U(-\theta-h) - dU(-\theta)\right]\left[b\varphi(\theta) + d\frac{d\varphi(\theta)}{d\theta}\right]d\theta + \\ -2\varphi(0)c_1 p\int_{-2h}^{0}\left[U(-\theta-2h) - dU(-h-\theta)\right]\varphi(\theta)d\theta + \\ +\int_{-h}^{0}\int_{-h}^{0} U(\theta-\eta)\left[b\varphi(\theta) + d\frac{d\varphi(\theta)}{d\theta}\right]\left[b\varphi(\eta) + d\frac{d\varphi(\eta)}{d\eta}\right]d\eta d\theta + \\ -2c_1 p\int_{-h}^{0}\int_{-2h}^{0} U(-h+\theta-\eta)\left[b\varphi(\theta) + d\frac{d\varphi(\theta)}{d\theta}\right]\varphi(\eta)d\theta d\eta + \\ +c_1^2 p^2\int_{-2h}^{0}\int_{-2h}^{0} U(\theta-\eta)\varphi(\theta)\varphi(\eta)d\theta d\eta \end{gathered} \tag{5.138}$$

The value of the performance index of quality (5.137) is equal to the value of the functional (5.138) for initial function φ

$$J = v(\varphi) \tag{5.139}$$

To obtain the value of the performance index of quality one needs a Lyapunov matrix $U(\xi)$ for $\xi \in [0,2h]$. In Chapter 5.2.5 was presented a method of determination of the Lyapunov matrix for a system with two delays.
System of equations (5.122) takes a form

$$\begin{bmatrix} \frac{d}{d\xi}U_1(\xi) \\ \frac{d}{d\xi}U_2(\xi) \\ \frac{d}{d\xi}Z_1(\xi) \\ \frac{d}{d\xi}Z_2(\xi) \end{bmatrix} = \mathscr{H} \begin{bmatrix} U_1(\xi) \\ U_2(\xi) \\ Z_1(\xi) \\ Z_2(\xi) \end{bmatrix} \tag{5.140}$$

where

$$\mathscr{H} = \begin{bmatrix} h_1 & -h_2 d & h_3 & h_2 \\ h_4 & h_5 & h_6 & h_2 d \\ -h_3 & -h_2 & -h_1 & h_2 d \\ -h_6 & -h_2 d & -h_4 & -h_5 \end{bmatrix}$$

where

$$h_1 = \frac{a-bd}{1-d^2},$$

$$h_2 = \frac{c_1 p}{1-d^2},$$

$$h_3 = \frac{b-ad}{1-d^2},$$

$$h_4 = \frac{b+ad-2bd^2}{1-d^2},$$

$$h_5 = \frac{a-ad^2-c_1pd^2}{1-d^2},$$

$$h_6 = \frac{bd-ad^2+c_1p-c_1pd^2}{1-d^2}$$

Initial conditions of system (5.140) one obtains solving the algebraic equation

$$Q \begin{bmatrix} U_1(0) \\ U_2(0) \\ Z_1(0) \\ Z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -w \end{bmatrix} \tag{5.141}$$

where

$$Q = \begin{bmatrix} \Phi_{11}(h) & \Phi_{12}(h)-1 & \Phi_{13}(h) & \Phi_{14}(h) \\ \Phi_{31}(h)-1 & \Phi_{32}(h) & \Phi_{33}(h) & \Phi_{34}(h) \\ \Phi_{41}(h) & \Phi_{42}(h) & \Phi_{43}(h)-1 & \Phi_{44}(h) \\ 2(a-bd) & -c_1pd & 2(b-ad)+c_1pd & 2c_1p \end{bmatrix} \tag{5.142}$$

Where $\Phi(\xi)$ is a fundamental matrix of solutions of equation (5.140).
We search for an optimal gain which minimize the index (5.137) for the initial function φ given by a formula

$$\varphi(\theta) = \begin{cases} x_0 & for\ \theta = 0 \\ 0 & for\ \theta \in [-2h, 0) \end{cases} \tag{5.143}$$

For the initial function φ given by the formula (5.143) the performance index of quality has a form

$$J = v(\varphi) = [(1+d^2)U_1(0) - 2dZ_1(0)]x_0^2 \tag{5.144}$$

Figure 5.7 shows the value of the index $J(p)$ for $x_0 = 1$, $w = 1$, $a = -1$, $b = -0.5$, $c_1 = -0.4$, $d = -0.6$ and $h = 1$. You can see that there exists a critical value of the gain p_{crit}. The system (5.136) is stable for gains less then critical one and unstable for gains greater then critical.

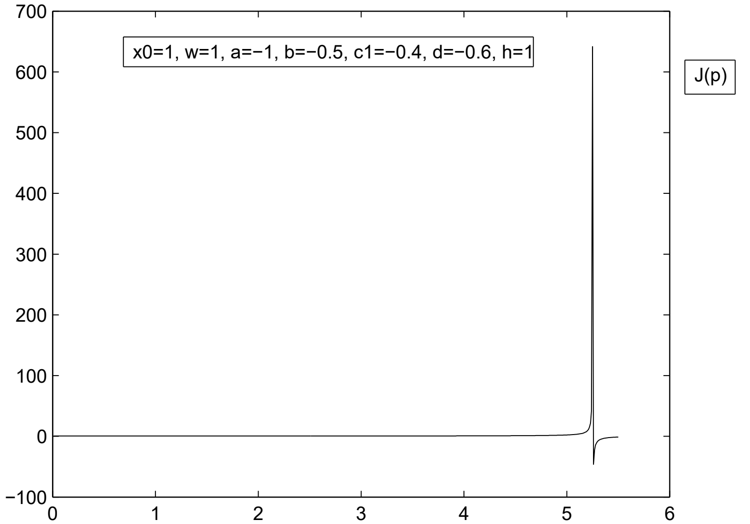

Fig. 5.7. Value of the index $J(p)$

Figure 5.8 shows the value of the index $J(p)$ for $x_0 = 1$, $w = 1$, $a = -1$, $b = -0.5$, $c_1 = -0.4$, $d = -0.6$ $h = 1$ and for p less then critical gain. You can see that the function $J(p)$ is convex and has a minimum.

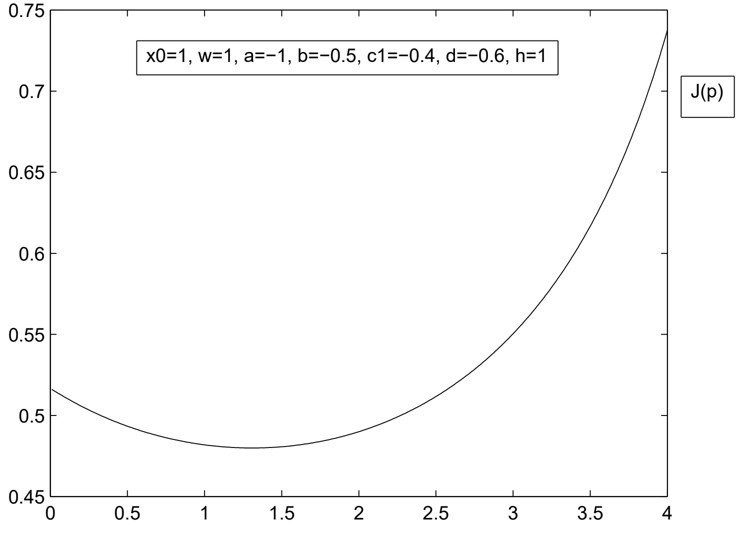

Fig. 5.8. Value of the index $J(p)$

Figure 5.9 shows graphs of functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$ and $Z_2(\xi)$ obtained with the Matlab code, for parameters of system (5.136) used in optimization process with $h = 1$ and for optimal gain.

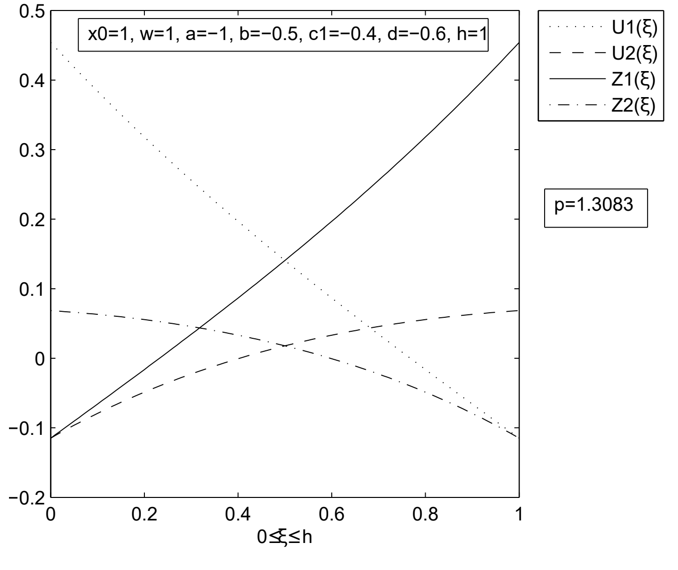

Fig. 5.9. Functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$ and $Z_2(\xi)$ for optimal gain

We search for an optimal gain of a P-controller which minimizes the index (5.144). Optimization results, obtained by means of Matlab function *fminsearch*, are given in Table 5.4. These results are obtained for $x_0 = 1$, $w = 1$, $a = -1$, $b = -0.5$, $c_1 = -0.4$ and $d = -0.6$.

Table 5.4

Optimization results

Delay h	Optimal gain	Index value	Critical gain
0.5	2.4291	0.4579	8.0
1.0	1.3083	0.4799	5.2
1.5	1.0145	0.4804	4.2
2.0	0.8803	0.4791	3.2
2.5	0.7986	0.4784	2.9
3.0	0.7433	0.4784	2.7

6 Conclusion

In the monograph was presented the method of determination of the Lyapunov functional for varies time delay systems and its applications to the parametric optimization problem to calculation of the quadratic performance index of quality, Integral of Squared Error (ISE). In the monograph were presented examples of parametric optimization problems for varies controllers i.e. P, I, PI, PD and for varies plants i.e. inertial system with one and two delays. An inertial system with delay (Küpfmüller model) is often used in practical applications so the obtained results can be useful. Interesting illustration of application of presented method to parametric optimization problem for separately excited D.C. motor angular velocity control system is presented in [10].
In monograph equations describing dynamics of time delay systems are given in a form of differential equations with time delay with respect to momentary state $x(t)$. We can reshape them to the state equation using the relation

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x_t(\theta)}{\partial \theta} \tag{6.1}$$

The procedure to obtain the state equation is presented below.
Let us consider a system with time delay whose dynamics is described by a functional-differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bx_t(-r) \\ x(t_0) = x_0 \\ x_{t_0} = \Phi \end{cases} \tag{6.2}$$

for $t \geq t_0$, where $x(t) \in \mathbb{R}^n$ is a momentary state, $\Phi \in L^2([-r,0),\mathbb{R}^n)$ is an initial function, x_t is a shifted restriction of $x(\cdot)$ to an interval $[t-r,t)$ and is given by a formula

$$x_t(\theta) := x(t+\theta) \tag{6.3}$$

for $t \geq t_0,\ \theta \in [-r,0)$

The state of system (6.2) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \tag{6.4}$$

for $t \geq t_0$, where $x(t) \in \mathbb{R}^n$ is a momentary state, $x_t \in L^2([-r,0),\mathbb{R}^n)$.
The state space is defined by a formula

$$X = \mathbb{R}^n \times L^2([-r,0),\mathbb{R}^n) \tag{6.5}$$

We compute the time derivative of (6.4)

$$\frac{dS(t)}{dt} = \begin{bmatrix} \dfrac{dx(t)}{dt} \\ \dfrac{\partial x_t}{\partial t} \end{bmatrix} = \begin{bmatrix} Ax(t) + Bx_t(-r) \\ \dfrac{\partial x_t}{\partial \theta} \end{bmatrix} = \mathscr{A}S(t)$$

for $t \geq t_0$, $\theta \in [-r,0)$.

$$S(t_0) = \begin{bmatrix} x_0 \\ \Phi \end{bmatrix} = S_0 \tag{6.6}$$

In such a way we obtained an *abstract initial-value problem*

$$\begin{cases} \dfrac{dS(t)}{dt} = \mathscr{A}S(t) \\ S(t_0) = S_0 \in \mathscr{D}(\mathscr{A}) \end{cases} \tag{6.7}$$

for $t \geq t_0$, where $\mathscr{D}(\mathscr{A})$ is a domain of operator $\mathscr{A}$

$$\mathscr{D}(\mathscr{A}) = \left\{ S(t) \in X : \quad \frac{dS(t)}{dt} \in X \ for\, t \geq t_0, \theta \in [-r,0) \right\} \tag{6.8}$$

The state space X is a Hilbert space with an inner product

$$< S_1, S_2 >= x_1^T x_2 + \int_{-r}^{0} \Phi_1^T(\theta)\Phi_2(\theta)d\theta \tag{6.9}$$

where $S_i = \begin{bmatrix} x_i \\ \Phi_i \end{bmatrix} \in X$ for $i = 1,2$.
Now will be presented the third method of determination of the Lyapunov functional.

Proposition 6.1. [32] *The solution of (6.7) is exponentially stable if and only if there exists a linear operator $\mathscr{H} = \mathscr{H}^\star$, defined on X, non-negatively definite i.e.*

$$< S(t), \mathscr{H}S(t) > \geq 0$$

for every $S(t) \in X$, $t \geq t_0$ is fixed *such that*

$$< \mathscr{A}S(t), \mathscr{H}S(t) > + < S(t), \mathscr{H}\mathscr{A}S(t) >= -x^T(t)Wx(t) \tag{6.10}$$

for $S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \in \mathscr{D}(\mathscr{A}) \subset X$, $t \geq t_0$ *is fixed, where* $W = W^T > 0$, $W \in \mathbb{R}^{n\times n}$ *is symmetric positively definite real matrix.*

The formula (6.10) is called *The Lyapunov operator equation* and enables us to determine a linear operator $\mathscr{H}$.
The formula

$$V(S(t)) =< S(t), \mathscr{H}S(t) > \tag{6.11}$$

for $t \geq t_0$, defines the quadratic Lyapunov functional.
There holds the relationship

$$J = \int_{t_0}^{\infty} x^T(t)Wx(t)dt =< S_0, \mathscr{H}S_0 > \tag{6.12}$$

for $S_0 = \begin{bmatrix} x_0 \\ \Phi \end{bmatrix}$.
This method gives the same results for system (6.2), as methods presented in the monograph, see [32] pages 146–148.
The Lyapunov matrices are also used in LQ problem for time delay systems to find the suboptimal control [82]. The optimal control is the solution of the Bellman type equation. In the monograph [68] the LQ problem for time delay systems is solved by means of variational method.

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