AGH University of Science & Technology
Faculty of Electrical Engineering, Automatics, Computer Science and Electronics

PhD Dissertation

Multi-Objective Portfolio Optimization by Mixed Integer Programming

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Rozprawa doktorska

Wielokryterialna optymalizacja portfelowa metodami programowania całkowitoliczbowego mieszanego

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List of Symbols

Notations for Mathematical Models M1 – M27.

Indices

i	=	historical time period, $i \in I = \{1, \ldots, m\}$ (i.e. day, week, month, etc.)
j	=	security, $j \in J = \{1, \ldots, n\}$
k	=	historical multi-period interval $k \in K = \{1, \ldots, t\}$ (i.e. year, quarter or month, etc) The set K is used in the mathematical model M22
l	=	historical successive investment period $l \in L = \{1, \ldots, t\}$ (i.e. year, quarter or month, etc) The set L is used in the mathematical model M23

Input parameters

α	=	confidence level - input parameter in selected problems List of the mathematical models, where α is the input parameter: M1, M4, M7, M8, M9, M11, M12, M16, M19, M22
$\beta_1, \beta_2, \beta_3, \lambda$	=	weights in the objective functions
γ	=	small positive value
$cov(r_i, r_j)$	=	matrix of covariance The matrix of covariance is the input parameter in the mathematical models: M3, M6, M13, M18, M21
f_1^{opt}	=	ideal solution (or chosen by a decision maker) value of Conditional Value-at-Risk ($CVaR$) The mathematical models, in which the input parameter f_1^{opt} is used, are: M8, M9, M16, M19
f_2^{opt}	=	ideal solution (or chosen by a decision maker) value of expected portfolio return The mathematical models, in which the input parameter f_2^{opt} is used, are: M7, M9, M10, M12, M13, M15, M16, M17, M18, M19, M20, M21, M23
f_3^{opt}	=	ideal solution (or chosen by a decision maker) value of confidence level The mathematical models, in which the input parameter f_3^{opt} is used, are: M11, M12, M17, M20

f_4^{opt} = ideal solution (or chosen by a decision maker) risk value in Markowitz portfolio

The mathematical models, in which the input parameter f_4^{opt} is used, are: M14, M15, M18, M21

f_5^{opt} = ideal solution (or chosen by a decision maker) value of number of assets (stocks) in optimal portfolio

The mathematical models, in which the input parameter f_5^{opt} is used, are: M19, M20, M21, M23

g = number of historical quotations in each successive investment period

This parameter is used only in the mathematical model M23

h = number of historical quotations in each multi-period intervals

This parameter is used only in the mathematical model M22

p_i = probability assigned to the occurrence of past realization i

r_{ij} = observed return of jth stock in ith time period

r^{Min} = minimum return observed in the market

VaR = return Value-at-Risk

List of the mathematical models, where VaR is the input parameter: M2, M5, M10, M11, M12, M17, M20, M22, M23

Variables

α = confidence level - variable in a selected problem

List of the mathematical models, where α is the decision variable: M2, M5, M10, M17, M20

R_i = tail return, i.e. the amount by which VaR exceeds return in scenario i

VaR = Value-at-Risk of portfolio return based on the α - percentile of return, i.e., in $100\alpha\%$ of historical portfolio realization, the outcome must be greater than VaR

List of the mathematical models, where VaR is the decision variable: M1, M4, M7, M8, M9, M16, M19

x_j = amount of capital invested in security j

y_i = 1 if return of portfolio in ith time period is over threshold VaR, 0 otherwise

z_j	$=$	1 if capital is invested in jth stock, 0 otherwise
δ	$=$	deviation from the reference solution
x_{ij}^k	$=$	percentage of capital invested in period i in security j of multi-period interval k This variable is used only in the mathematical model M22
y_i^k	$=$	1 if return of portfolio in period i of multi-period interval k is over threshold VaR, 0 otherwise This variable is used only in the mathematical model M22
x_{jl}	$=$	percentage of capital invested in successive investment period l in security j This variable is used only in the mathematical model M23
x_{jl}^{buy}	$=$	percentage of capital invested in successive investment period l for bought security j This variable is used only in the mathematical model M23
x_{jl}^{sell}	$=$	percentage of capital reached in successive investment period l by selling security j This variable is used only in the mathematical model M23
y_{il}	$=$	1 if return of portfolio in period i of successive investment period l is over threshold VaR, 0 otherwise This variable is used only in the mathematical model M23
α_l	$=$	confidence level for successive investment period l This variable is used only in the mathematical model M23
z_{jl}	$=$	1 if in successive investment period l capital is invested in security j 0 otherwise This variable is used only in the mathematical model M23

Notations for Mathematical Models M28 – M30.

Indices

i = worker, $i \in I = \{1, \ldots, m\}$

j = supporting service hospital department, $j \in J = \{1, \ldots, n\}$

k = type of supporting service job, $k \in K = \{1, \ldots, q\}$

Input parameters

$\bar{c_{ik}}$ = cost of assignment of a worker i to job k (i.e. monthly salary)

$\bar{C_j}$ = maximal monthly budget for salaries in a department j

$\bar{e_k}$ = size of permanent (partial or full time) employment for job k (i.e. $\bar{e_k}$= 0.25 or 0.50 or 0.75 or 1.00)

$\bar{E_j}$ = maximal number of permanent employments in a department j

$\bar{h_{jk}}$ = minimal number of permanent employments for job k in a department j

$\bar{\beta_i}$ = weight of the objective functions $\bar{f_i}, i = 1, 2, 3$

γ = small positive value

$\bar{f_1^{opt}}$ = ideal solution value of number of workers selected for an assignment to any job in any department

$\bar{f_2^{opt}}$ = ideal solution value of operational costs of the supporting services

$\bar{f_3^{opt}}$ = ideal solution value of number of permanent employments for all jobs in all departments

Variables

$\bar{x_{ijk}}$ = 1 if worker i is assigned to job k in department j, 0 otherwise

$\bar{y_i}$ = 1 if worker i is assigned to any job in any department, 0 otherwise

$\bar{g_{jk}}$ = number of permanent employments for job k in department j

$\bar{z}$ = total number of workers assigned to any job in any department

δ = deviation from the reference solution

List of Abbreviations

$B - \& - B$ *nodes*	=	the number of searched nodes in the branch and bound tree until presented solution.
CPU	=	Central Processing Unit
$CVaR$	=	Conditional Value-at-Risk
GAP	=	percentage difference between obtained solution and the best solution calculated by the CPLEX solver
IP	=	Integer Programming
LP	=	Linear Programming
MIP	=	Mixed Integer Programming
$M1$	=	Conditional Value-at-Risk Bi-Criteria Portfolio Model with Weighting Approach
$M2$	=	Value-at-Risk Bi-Objective Portfolio Model with Weighting Approach
$M3$	=	Bi-Objective Markowitz Portfolio Model with Weighting Approach
$M4$	=	Conditional Value-at-Risk Triple-Criteria Portfolio Model with Weighting Approach
$M5$	=	Value-at-Risk Triple-Objective Portfolio Model with Weighting Approach
$M6$	=	Triple-Objective Markowitz Portfolio Model with Weighting Approach
$M7$	=	Portfolio Model with Conditional Value-at-Risk with Risk Measure as Primary Objective by Lexicographic Approach
$M8$	=	Portfolio Model with Conditional Value-at-Risk with Portfolio Expected Return as Primary Objective by Lexicographic Approach
$M9$	=	Portfolio Model with Conditional Value-at-Risk with Number of Assets in Optimal Portfolio as Primary Objective by Lexicographic Approach
$M10$	=	Portfolio Model with Value-at-Risk with Risk Measure as Primary Objective by Lexicographic Approach
$M11$	=	Portfolio Model with Value-at-Risk with Portfolio Expected Return as Primary Objective by Lexicographic Approach
$M12$	=	Portfolio Model with Value-at-Risk with Number of Assets in Optimal Portfolio as Primary Objective by Lexicographic Approach

$M13$ = Portfolio Model based on Classical Markowitz Method with Risk Measure as Primary Objective by Lexicographic Approach

$M14$ = Portfolio Model based on Classical Markowitz Method with Portfolio Expected Return as Primary Objective by Lexicographic Approach

$M15$ = Portfolio Model based on Classical Markowitz Method with Number of Assets in Optimal Portfolio as Primary Objective by Lexicographic Approach

$M16$ = Bi-Objective Portfolio Model with Conditional Value-at-Risk by Reference Point Approach

$M17$ = Bi-Objective Portfolio Model with Value-at-Risk by Reference Point Approach

$M18$ = Bi-Objective Markowitz Portfolio Model by Reference Point Approach

$M19$ = Triple-Objective Portfolio Model with Conditional Value-at-Risk and Number of Assets in Optimal Portfolio as Third Objective by Reference Point Approach

$M20$ = Triple-Objective Portfolio Model with Value-at-Risk and Number of Assets in Optimal Portfolio as Third Objective by Reference Point Approach

$M21$ = Triple-Objective Markowitz Portfolio Model and Number of Assets in Optimal Portfolio as Third Objective by Reference Point Approach

$M22$ = Multi-Period Portfolio Model with Weighting Approach

$M23$ = Weighted-Sum Multi-Objective Portfolio Optimization Model with Multi-Period Approach

$M24$ = Bi-Criteria Portfolio Model with Tail Return

$M25$ = Value-at-Risk with Bi-Objective Portfolio Model with Relaxed Constraint for Invested Amount of Capital

$M26$ = Value-at-Risk Triple-Objective Portfolio Model with Amount of Capital as Additional Criterion

$M27$ = Value-at-Risk Four Criterion Portfolio Model with Amount of Capital as Additional Objective

$M28$ = Single-Criteria Optimization Model for Assignment of Supporting Services in Health Care

$M29$	=	Single-Criteria Optimization Model for Assignment of Supporting Services in Health Care with alternative constraint formulation
$M30$	=	Triple-Objective Optimization Model for Assignment of Supporting Services in Health Care by Reference Point Approach
QP	=	Quadratic Programming
Simplex iteration	=	the number of simplex iterations until presented solution
VaR	=	Value-at-Risk

I would like to express my thanks to Professor Jan Tadeusz Duda for his valuable comments, review directions on an earlier version of the dissertation, and for reading the whole dissertation carefully and providing constructive comments which helped to substantially improve this PhD dissertation.

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Summary

MULTI-OBJECTIVE PORTFOLIO OPTIMIZATION BY MIXED INTEGER PROGRAMMING

Summary: In this PhD dissertation the mathematical programming methods of operations research for multi-criteria optimization are presented. The PhD dissertation deals with the problem of selection of methods and numerical tools for solving portfolio optimization problems with different objectives. In particular, the research efforts were concentrated on mixed integer programming formulations. The need for solving multi-objective portfolio optimization models by mixed integer programming can be illustrated for the portfolio models with Value-at-Risk (VaR) as a risk measure, as well as, when the number of assets (investment ventures) is one of the optimality criteria. An alternative, multi-objective portfolio optimization problems is formulated with Conditional Value-at-Risk ($CVaR$) as a risk measure or with symmetric measure of risk - covariance (variance) of return - as in Markowitz portfolio.

The portfolio models with $CVaR$ and with covariance (variance) of historical return were being solved with the use of mathematical programming with the continuous variables. The proposed multi-objective portfolio models are constructed with the expected return as a performance measure and the expected worst-case return as a risk measure, using Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$). These measures allow the evaluation of worst-case return and shaping of the resulting return distribution through the selection of the optimal portfolio. The mathematical programming models are constructed and solved using weighting, lexicographic and reference point approach. The presented portfolio models are single-, bi- and triple-objectives and the optimization criteria considered are risk, return and number of stocks.

The main research problem considered in this Ph.D. dissertation is the way for finding the best multi-objective portfolio formulation with risk. The additional research problem is to find the relation between the optimization results with Value-at-Risk solved by mixed integer programming and the results obtained with the use of linear and quadratic programming portfolio models (Conditional Value-at-Risk, Markowitz).

Computational experiments have been conducted for multi-criteria portfolio models of stock exchange investments. The input data for computations consist of historical daily returns of stocks quoted on Warsaw Stock Exchange. The number of selected securities for input data varies from 46 to 240 assets. The historical stocks quotations come from the period from March 10th, 1997 to February 2nd, 2009. This time period

includes data from the increase of stock exchange quotations, as well as the economic crisis period. The considered number of data in historical time series is from 500 to 3000 days with assets quoted each day in the whole historical horizon. The portfolios were optimized in an increased time window, which was helpful in evaluating the results of optimization (time-varying optimal portfolio).

The multi-criteria portfolio optimization models with Conditional Value-at-Risk ($CVaR$) as a risk measure can be used to support on-line stock market investments, since the computational times required to find the optimal solution is relatively short, regardless of the size of the input data. The presented models provide a decision maker with a tool for evaluating the relationship between expected and worst-case returns.

The results obtained from computational experiments proved, that multi-objective portfolio optimization models with Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) could be used to shape the distribution of portfolio returns in a favorable way for a decision maker. The portfolios obtained with both methods (mixed-integer or linear programming) are often similar results, which shows their capability of solving the corresponding problems. It means that a suboptimal portfolio for the integer program with Value-at-Risk (VaR) as optimality criterion can be found by solving the corresponding linear program for the portfolio problem with Conditional Value-at-Risk ($CVaR$) as an optimality criterion. The proposed scenario-based portfolio optimization problems under uncertainty, formulated as a single- or multi-objective mixed integer program were solved using commercially available software (AMPL/CPLEX) for mixed integer programming.

In addition to the multi-objective approach for portfolio optimization of securities (e.g. stocks) from stock exchanges presented in this dissertation, the selected multi-objective mixed integer programming models are shown for supporting services in medical care institutions, based on an assignment problem.

Key words: Multi-Criteria Decision Making, Mathematical Programming, Mixed Integer Programming, Linear Programming, Quadratic Programming, Portfolio Optimization, Conditional Value-at-Risk, Value-at-Risk, Weighting Approach, Lexicographic Approach, Reference Point Method.

Mathematics Subject Classification: 90C05, 90C11, 90C20, 90C29, 90C90, 91G10.

Streszczenie

WIELOKRYTERIALNA OPTYMALIZACJA PORTFELOWA METODAMI PROGRAMOWANIA CAŁKOWITOLICZBOWEGO MIESZANEGO

Streszczenie: W rozprawie doktorskiej przedstawione zostały metody badań operacyjnych programowania matematycznego dla zadań wielokryterialnej optymalizacji portfelowej. W rozprawie podjęto problem doboru metod i narzędzi numerycznych do rozwiązywania zadań optymalizacji portfelowej przy różnych wskaźnikach jakości. W szczególności skoncentrowano się na zadaniach wymagających metod programowania całkowitoliczbowego mieszanego. Potrzeba rozwiązywania takich zadań powstaje w przypadku, gdy celem optymalizacji jest minimalizacja wartości zagrożonej zwrotu (portfele z miarą Value-at-Risk), a także kiedy celem optymalizacji jest znalezienie optymalnej liczby spółek (przedsięwzięć inwestycyjnych) w poszukiwanym portfelu. Jako alternatywne formułowano zadania minimalizacji warunkowej wartości zagrożonej zwrotu (portfele z miarą Conditional Value-at-Risk) oraz minimalizację ryzyka symetrycznego wyrażonego kowariancją (wariancją) stopy zwrotu z portfela (portfele Markowitza), które rozwiązuje się znacznie szybciej metodą programowania na liczbach rzeczywistych.

Zaakcentowano wielokryterialność faktycznych zadań optymalizacji portfelowej. Powierzchnie kompromisu wyszukiwano za pomocą trzech metod: metodą ważonej funkcji celu, metodą leksykograficzną oraz metodą punktów referencyjnych.

Problemem badawczym podjętym w rozprawie jest sposób najlepszego wyznaczenia portfela (przedsięwzięć - inwestycji) w warunkach ryzyka przy uwzględnieniu wielu kryteriów optymalności. Problemem badawczym jest także naświetlenie relacji pomiędzy efektami optymalizacji Value-at-Risk uzyskanymi metodami programowania mieszanego, wobec rozwiązań uzyskanych znacznie szybciej metodami programowania liniowego i kwadratowego na liczbach rzeczywistych (Conditional Value-at-Risk, Markowitz).

Analizy numeryczne zostały przeprowadzone na przykładzie zadań optymalizacji portfela inwestycji giełdowych. Wykorzystano dane o stopach zwrotu akcji notowanych na Giełdzie Papierów Wartościowych w Warszawie. Rzeczywiste dane z giełdy papierów wartościowych obejmują zarówno okresy koniunktury jak i dekoniunktury na rynku. Szeregi czasowe stóp zwrotu dotyczą od 46 do 240 spółek. Dane historyczne obejmują okresy czasu od 10go marca 1997 roku do 2go lutego 2009 roku. Liczba danych w poszczególnych szeregach czasowych waha się od 500 do 3000. Portfele były optymalizowane w rozszerzanym oknie, co pozwoliło ocenić niepewność wyników optymalizacji (zmienność portfeli optymalnych w czasie).

Obliczenia wykazały, że wielokryterialna optymalizacja portfelowa według ustalonego ryzyka przekroczenia wartości zagrożonej zwrotu (Value-at-Risk) oraz warunkowej wartości zagrożonej zwrotu (Conditional Value-at-Risk) pozwalają kształtować rozkład zwrotów z portfela w sposób korzystny dla decydenta. Portfele uzyskane dwiema metodami są często zbliżone. Wskazuje to na możliwość szybkiego uzyskania rozwiązań suboptymalnych, w sensie ryzyka przekroczenia wartości zagrożonej zwrotu, metodami programowania liniowego.

Wykazano użyteczność podejścia wielokryterialnego, jako metody wspomagania decyzji. Pokazano, że czas obliczeń zależy w nieznacznym stopniu od liczby analizowanych szeregów czasowych i ich długości. Sformułowane zadania optymalizacji portfelowej z dwoma oraz trzema kryteriami decyzyjnymi umożliwiają znajdowanie rozwiązań optymalnych w relatywnie krótkim czasie i mogą stanowić narzędzie wspomagania decyzji operacyjnych.

W rozprawie doktorskiej zamieszczono dodatkowo rozdział prezentujący przykład zastosowania metod optymalizacji wielokryterialnej w zadaniu przydziału pracowników w działach szpitala zajmujących się działalnością pomocniczą. Kryteriami decyzyjnymi w tym zadaniu były koszty działalności, liczby zatrudnionych pracowników oraz liczba zajmowanych etatów. Zadanie zostało rozwiązane metodami programowania całkowitoliczbowego mieszanego.

Słowa kluczowe: Wielokryterialne wspomaganie decyzji, programowanie matematyczne, optymalizacja portfelowa, Conditional Value-at-Risk (warunkowa wartość zagrożona zwrotu), Value-at-Risk (wartość zagrożona zwrotu), metoda ważonej funkcji celu, metoda leksykograficzna, metoda punktów referencyjnych.

Preface

In this PhD dissertation the mathematical programming methods of operations research for multi-criteria optimization are presented. The PhD dissertation deals with the problem of selection of methods and numerical tools for solving portfolio optimization problems with different objectives. In particular, the research efforts were concentrated on mixed integer programming formulations. The need for solving multi-objective portfolio optimization models by mixed integer programming can be illustrated for the portfolio models with Value-at-Risk (VaR) as a risk measure, as well as, when the number of assets (investment ventures) is one of the optimality criteria. An alternative, multi-objective portfolio optimization problems is formulated with Conditional Value-at-Risk ($CVaR$) as a risk measure or with symmetric measure of risk - covariance (variance) of return - as in Markowitz portfolio.

The portfolio models with $CVaR$ and with covariance (variance) of historical return were being solved with the use of mathematical programming with the continuous variables. The proposed multi-objective portfolio models are constructed with the expected return as a performance measure and the expected worst-case return as a risk measure, using Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$). These measures allow the evaluation of worst-case return and shaping of the resulting return distribution through the selection of the optimal portfolio. The mathematical programming models are constructed and solved using weighting, lexicographic and reference point approach. The presented portfolio models are single-, bi- and triple-objectives and the optimization criteria considered are risk, return and number of stocks.

The main research problem considered in this Ph.D. dissertation is the way for finding the best multi-objective portfolio formulation with risk. The additional research problem is to find the relation between the optimization results with Value-at-Risk solved by mixed integer programming and the results obtained with the use of linear and quadratic programming portfolio models (Conditional Value-at-Risk, Markowitz).

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The multi-criteria portfolio optimization models with Conditional Value-at-Risk ($CVaR$) as a risk measure can be used to support on-line stock market investments, since the computational times required to find the optimal solution is relatively short, regardless of the size of the input data. The presented models provide a decision maker with a tool for evaluating the relationship between expected and worst-case returns.

The results obtained from computational experiments proved, that multi-objective portfolio optimization models with Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) could be used to shape the distribution of portfolio returns in a favorable way for a decision maker. The portfolios obtained with both methods (mixed-integer or linear programming) are often similar results, which shows their capability of solving the corresponding problems. It means that a suboptimal portfolio for the integer program with Value-at-Risk (VaR) as optimality criterion can be found by solving the corresponding linear program for the portfolio problem with Conditional Value-at-Risk ($CVaR$) as an optimality criterion. The proposed scenario-based portfolio optimization problems under uncertainty, formulated as a single- or multi-objective mixed integer program were solved using commercially available software (AMPL/CPLEX) for mixed integer programming.

The nature of the problem is to find a compromise between the construction of objectives, constraints and decision variables in a portfolio and the problem complexity of the implemented mathematical models. There is always a trade off between computational time and the size of an input data, as well as the type of mathematical programming formulation (linear or mixed integer).

The computational results obtained by modeling the decision criteria (e.g. lexicographically choosing one objective function with the highest priority) in constructed multi-objective portfolio optimization models, could be used by a decision maker for evaluation of his/her investment strategy. It is easy to compare obtained optimal (ideal) solution values of selected objectives with a real investment situation in a stock market.

The portfolio optimization models with $CVaR$ could be used for supporting on-line stock market investments, since computational times required for finding optimal solutions are relatively short, regardless of the size of input data for computations (e.g. more than 200 stocks with 3000 quotations).

In addition to the multi-objective approach for portfolio optimization of securities (e.g. stocks) from stock exchanges presented in this dissertation, the selected multi-objective mixed integer programming models are shown for supporting services in medical care institutions, based on an assignment problem.

Several publications on multi-objective portfolio optimization have been written by the author of this PhD dissertation (see e.g. [101, 102, 103, 104, 105, 109, 107, 108, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 123, 124, 125, 126, 127, 128]).

Scope

Chapter 1 describes state-of-the-art of portfolio formulations, including formal basis for implemented multi-criteria methods and numerical tools in portfolio optimization. Literature review on risk measures used in portfolio optimization is also placed in this chapter. The proposed methodological framework for the weighting, lexicographic and reference point approaches of multi-criteria portfolio selection process are presented. The historical and theoretical background of mathematical programming methods of multi-criteria optimization is also explained. A list of available computer software packages typically used to solve mathematical programming, (especially linear and mixed-integer problems) is added. Moreover, short descriptions and analysis of input data sets used for computational experiments are presented.

Chapter 2 shows weighting approach to multi-objective portfolio optimization models. The portfolio models are constructed using different risk measures. Bi-Objective models with Conditional Value-at-Risk ($CVaR$), Value-at-Risk (VaR) and covariance matrix as risk measures are presented in the first part of this chapter. In the second part, models are formulated as triple-objective portfolio optimization with maximization or minimization of number of securities (e.g. stocks) in optimal portfolio as a third criterion.

Chapter 3 presents lexicographic approach to multi-objective portfolio optimization models. The first part of this chapter presents portfolio models with two optimization criteria: risk and return. In the second part, the auxiliary criterion is the number of securities (e.g. stocks). The models are solved lexicographically.

Chapter 4 contains reference point approach to multi-objective portfolio optimization models. Considered multi-objective portfolio models presented in this chapter are solved using ideal values for each objective and minimizing a distance from obtained to ideal value for each objective. Presented models are constructed for bi- and triple-objective portfolio optimizations.

Chapter 5 shows selected examples of multi-period portfolio models.

Chapter 6 presents some alternative portfolio formulations.

Chapter 7 deals with selected multi-objective mixed integer programming models for supporting services in medical care institutions, based on assignment problem, together with some computational examples.

Chapter 8 presents the results of computational experiments with the proposed optimization models.

Chapter 9 contains final conclusion and comments on future research directions.

In the additional chapter - appendix - some more computational results are presented.

Acknowledgements

I would like to express my thanks to Professor Jan Tadeusz Duda for his valuable comments, review directions on an earlier version of the dissertation, and for reading the whole dissertation carefully and providing constructive comments which helped to substantially improve this PhD dissertation.

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Aims and Theses

Goals

The purpose of the study is to update and develop new decision models for a multi-objective portfolio optimization under uncertainty, using the modern probability management approach. Probability management shifts the focus away from single numbers to probability distributions, which is a prerequisite for the effective management of risk, real portfolios, real options, etc. The purpose of the study is to update the knowledge and to elaborate new methods in portfolio optimization, modeling uncertainty, time series analysis, and stochastic optimization in financial engineering and new methods of operation research, especially in multi-objective optimization approaches.

The portfolio problem objective is to allocate wealth among different assets to maximize a set of performance functions.

The portfolio problem is formulated as an optimization problem involving two criteria: the reward of the portfolio that should be maximized, and the risk of the portfolio that should be minimized. In the presence of two criteria there is not a single optimal solution (portfolio structure), but a set of optimal portfolios, the so-called efficient portfolios, which trade-off between risk and return.

In the classical portfolio approach, future returns are random variables that can be controlled by two parameters: the portfolio efficiency, which is measured by the expected return, while risk is calculated by the standard deviation of returns. As a result the classical problem is formulated as a quadratic programming task with continuous variables and some side constraints.

In this dissertation bi-objective portfolio models are constructed with the expected return as a performance measure and the expected worst-case return as a risk measure.

Weighted-sum, lexicographic and reference point approaches have been implemented to find solutions for bi- and triple-objective portfolio optimization problems. The first objective function defines risk of portfolio venture, this objective aims ta minimization of risk subject to specific constraints. The second objective is to maximize portfolio expected return. The third objective function is the number of securities in optimal portfolio to be maximized or minimized (According to a decision maker preferences.).

Computational experiments with the linear and mixed integer programming approach, modeled after a real data from the Warsaw Stock Exchange were performed.

The portfolio optimization problem is formulated as a multi-objective mixed integer linear program, which allows commercially available software (e.g. AMPL/CPLEX) to be applied for solving medium size, yet practical instances.

A decision maker's preferences are an important factor to be considered in a portfolio formulation. If a portfolio's loss is under Value-at-Risk (VaR) (e.g. a risk of bancruptcy), then mixed integer programming Value-at-Risk (VaR) portfolio model with risk probability calculated as $1 - \alpha$ needs to be considered (see e.g. **M2**, **M5**, **M10**, **M11**, **M12**, **M17**, **M20**, **M22**, **M23**).

In case when a decision maker is willing to minimize value of worst expected return, a portfolio model should be formulated with Conditional Value-at-Risk ($CVaR$) as a risk measure (see e.g. **M1**, **M4**, **M7**, **M8**, **M9**, **M16**, **M19**).

When a decision maker is trying to find a solution which will satisfy both sides of a transaction (for instance an investor and a market), in that case Markowitz portfolio model with symmetric risk measure should be considered (see e.g. **M3**, **M6**, **M13**, **M14**, **M15**, **M18**, **M21**).

Theses

This Ph.D. dissertation is going to prove the theses presented below.

Thesis I: Real life decision making problems in the portfolio selection can be solved using formal methods of multi-objective optimization with percentile risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) or with covariance matrix (the Markowitz model).

Thesis II: The optimization models formulated by mixed integer programming can be effectively implemented in the decision support systems for the multi-objective portfolio optimization, in which variance of return (cost) from the risky ventures is replaced with Value-at-Risk (VaR) or Conditional Value-at-Risk ($CVaR$) of return (cost).

Thesis III: The multi-criteria portfolio optimization models with Conditional Value-at-Risk ($CVaR$) as a risk measure can be used to support on-line stock market investments, since the computational times required to find optimal solution is relatively short, regardless of the size of input data.

Thesis IV: Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) allow for the evaluation of worst-case return (cost) and for shaping of the resulting return (cost) distribution through the selection of optimal portfolio.

Thesis V: A suboptimal portfolio for the integer program with Value-at-Risk (VaR) as optimality criterion can be found by solving the corresponding linear program for the portfolio problem with Conditional Value-at-Risk ($CVaR$) as an optimality criterion.

Thesis VI: The scenario-based portfolio optimization problem under uncertainty, formulated as a single- or multi-objective mixed integer program can be solved using commercially available software for mixed integer programming.

Thesis VII: A multi-objective portfolio problem with minimum number of assets as an auxiliary criterion is a mixed integer program.

Thesis VIII: The proposed models provide the decision maker with a simple tool for evaluating the relationship between expected and worst-case returns (costs).

The theses presented in this dissertation will be proved by the literature review, the portfolio models formulations and the results of computational experiments.

Tezy rozprawy

Teza I: Rzeczywiste problemy optymalizacji portfelowej mogą być rozwiązywane przy użyciu formalnych metod optymalizacji wielokryterialnej z kwantylowymi miarami ryzyka VaR (wartość zagrożona zwrotu) oraz CVaR (warunkowa wartość zagrożona zwrotu) lub z symetryczną miarą ryzyka wyrażoną macierzą kowariancji – wariancją - zwrotów (Portfel Markowitz'a).

Teza II: Modele optymalizacji sformułowane metodami programowania całkowitoliczbowego mieszanego mogą być efektywnie zastosowane w systemach wspomagania decyzji dla wielokryterialnej optymalizacji portfelowej, w której wariancja zwrotu (straty) dla ryzykownych przedsięwzięć (projektów) może zostać zastąpiona przez VaR lub CVaR.

Teza III: Wielokryterialne modele optymalizacji portfelowej z miarą ryzyka CVaR mogą być wykorzystywane do podejmowania decyzji inwestycyjnych on-line, gdyż czasy obliczeń wymagane do znalezienia rozwiązań optymalnych są wystarczająco krótkie, bez względu na rozmiar rozwiązywanych zadań optymalizacji portfelowej.

Teza IV: Miary ryzyka VaR oraz CVaR pozwalają na ocenę zwrotu (straty) w najgorszym przypadku oraz kształtowanie rozkładu zwrotów (strat) poprzez wybór portfela optymalnego.

Teza V: Portfel suboptymalny względem miary ryzyka VaR dla modelu programowania całkowitoliczbowego można wyznaczyć przez rozwiązywanie odpowiedniego zadania programowania liniowego z miarą ryzyka CVaR.

Teza VI: Problem optymalizacji portfelowej w warunkach niepewności (zawierający miarę ryzyka jako jedno z kryteriów optymalizacji) dla różnych scenariuszy, sformułowany jako jedno- lub wielokryterialne zadanie programowania całkowitoliczbowego mieszanego można rozwiązać stosując komercyjne pakiety obliczeniowe dla programowania całkowitoliczbowego mieszanego.

Teza VII: Wielokryterialne zadania optymalizacji portfelowej z minimalną liczbą spółek (przedsięwzięć inwestycyjnych) jako pomocnicze kryterium optymalizacji jest zadaniem programowania całkowitoliczbowego mieszanego.

Teza VIII: Zaproponowane w pracy doktorskiej modele stanowią dla decydenta wygodne narzędzie do oceny relacji pomiędzy oczekiwaną a krytyczną wartością zwrotu (straty).

Chapter 1

Formal Basis and Numerical Tools

1.1 Portfolio Formulations

The development of new techniques in operational research, as well as the progress in computer and information technologies, has given rise to new approaches for modeling the problem for portfolio selection. The multi-criteria decision making provides a solid methodological basis for resolving the inherent multi-criteria nature of the problem. The multi-dimensional nature of the portfolio selection problem has been emphasized by many researchers, from the fields of financial engineering and multi-criteria decision making (see e.g. White, 1990; Spronk and Hallerbach, 1997; Steuer and Na, 2003; Steuer et al., 2005, 2006a, 2006b, 2007a, 2007b; Xidonas and Psarras, 2008; Zeleny, 1981; Zopounidis et al., 1998; Zopounidis, 1999; Zopounidis and Doumpos, 2002 [151, 137, 140, 141, 142, 143, 144, 145, 155, 162, 164, 165, 166]).

Research activity regarding the more specific level of applying the multi-objective optimization approaches in the field of portfolio selection is the most representative in the studies of Mansini et al., 2003b, 2007; Ehrgott et al., 2004; Ehrgott and Wiecek, 2005; Ogryczak, 2000; Zopounidis et al., 1998; Wierzbicki, 1977 [78, 79, 42, 43, 91, 164, 153].

The portfolio management process can be divided into three fundamental phases: planning, execution and feedback (see e.g. Maginn et al., 2007; Esch et al., 2005; Reilly and Brown, 2005; Xidonas et al., 2008 [75, 44, 96, 156]). In the planning phase, investment objectives and policies are formulated, capital market expectations are formed and strategic asset allocations are established. In the execution phase, the decision maker constructs the portfolio and integrates the investment strategies with capital market expectations to select the specific assets for the portfolio. In the feedback phase, the decision maker monitors and evaluates the portfolio compared with the plan.

The portfolio problem, which involves computing the proportion of the initial budget that should be allocated among the available securities, is at the core of the field of financial management. A fundamental answer to this problem was given by Markowitz (1952, 1997 [80, 81]) who proposed the mean-variance model which laid the basis of modern portfolio theory. In Markowitz's approach the problem is formulated as an optimization problem involving two criteria: the reward of portfolio, which is measured by the mean or expected value of return that should be maximized, and the risk of the portfolio, which is measured by the variance of return that should be minimized. In the presence of two criteria there is not a single optimal solution (portfolio structure), but a set of optimal portfolios, the so-called efficient portfolios, which trade-off between risk and return. Since the mean-variance theory of Markowitz, an enormous amount of papers have been published extending or modifying the basic model in three directions. The first path goes to simplification of the type and the amount of input data (see e.g. Bana and Soares, 2004; Benati and Rizzi, 2009; Bertsimas and Pachamanova, 2008; Brennan, 1975; Duda et al., 2011; Feinstein and Thapa, 1993; Zopounidis et al., 1998 [11, 17, 19, 25, 40, 45, 164]). The second direction concentrates on the introduction of an alternative measure of risk (e.g. Angelelli et al., 2007; Gaivoronski and Pflug, 2005; Konno et al., 1993; Lin, 2009; Ma and Wong, 2010; Michalowski and Ogryczak, 2001; Natarajan et al., 2009; [7, 51, 66, 71, 74, 87, 88]). Finally, the third involves the incorporation of the additional criteria and/or constraints (see e.g. Anagnostopoulos and Mamanis, 2010; Li and Xu, 2009; Martel et al., 1988; Bouri et al., 2002; Gaivoronski et al., 2005, Hamacher et al., 2010; Perez et al., 2007; Steuer et al., 2005; Xidonas et al., 2010 [6, 70, 83, 22, 52, 58, 93, 141, 157]).

The overall process of selecting a portfolio is divided into two stages (Markowitz, 1952 [80]). The first stage starts with observation, experience and ends with beliefs about the future performances of available securities. The second stage starts with relevant beliefs about future performances and ends with the choice of portfolio. One type of rule concerning choice of portfolio is that the investor should maximize the capitalized value of future returns. A decision maker places all his funds in the security with the greatest discounted value. Investor diversifies his founds among all those securities which give maximum expected return. If two or more securities have the same value, then any of these or any combination of these is as good as any other. However, the portfolio with maximum expected return is not necessarily the one with minimum risk. The law of large numbers (LLN) will insure that the actual yield of the portfolio will be almost the same as the expected yield. The size of input portfolio for computations is also important. Considered number of securities (stocks) taken as an input data for computation is often at least ten or more (see e.g. Guerard, 2010; Li and Xu, 2009; Mavralexakis et al., 2011; Salo et al., 2011 [55, 70, 76, 99]). The selection of stocks to an input data could be done by many ways. For example by taking all securities quoted each day, during the whole historical period or choosing some of them, for instance only the stocks from banking sector or it could be defined by a decision maker.

There is a rate at which the investor can gain expected return by taking on risk measure, or reducing risk by giving up expected return (Ogryczak, 2000, [91]). In the classical Markowitz model future returns are random variables that can be controlled by the two parameters: a portfolio's efficiency calculated by the expectation, and a risk, which is measured with variance. The classical problem is formulated as a quadratic program with continuous variables and some side constraints. Bai et al. (2009a, 2009b [9, 10]) have developed a new bootstrap-corrected estimator of the optimal return for the Markowitz mean-variance optimization. Markowitz and van Dijk (2003 [82]) find that under certain conditions, the single-period mean-variance model provides a good approximation to multi-period expected utility maximization.

Although the original Markowitz model forms a quadratic programming problem, following Sharpe (1971 [133]), many attempts have been made to linearize the portfolio optimization procedure (for instance Speranza, 1993 [136]). The linear program solvability is very important for applications to real-life financial and other decisions where the constructed portfolios have to meet numerous side constraints. The examples of them are minimum transaction lots, transaction costs or mutual funds characteristics etc. The introduction of these features leads to mixed integer programming problems.

For some basic investment decision-making approaches, the decision maker may be restricted to choosing only one of a discrete number of alternatives. For other scenarios, a diversified portfolio comprised of a convex combination of two or more alternatives may be feasible and will often better balance risk and return. Sharpe (1971, 1999 [133, 134]) stated that "*if the essence of the portfolio analysis problem could be adequately captured in a form suitable for linear programming methods, the prospect for application would be greatly enhanced*". Linear programming efforts a decision maker the opportunity to determine an optimal balance between risk and return for modeling portfolio optimization problems with diversification among alternatives.

There is a vast literature on portfolio selection devoted to the balancing of risk and return in financial markets. The most celebrated of these (as it was previously written) is the approach of Markowitz (1952, 1997 [80, 81]) where a quadratic mean-variance model with risk measured by the covariance matrix of returns was developed. Konno and Yamazaki (1991 [65]) noted that the derivation of the covariance matrix can be cumbersome, attempting to solve a quadratic model has computational limitations in practice, and the optimal solution may consist of purchasing a large number of securities. They suggest employing linear objectives to alleviate these computational limitations. In spite of the fact that Sharpe (1963, 1999 [131, 134]) developed a methodology for practical solution of the quadratic objective, many approaches have been taken to linearize the model. Sharpe (1967, 1971, 1999 [132, 133, 134]) and Stone (1973 [146]) both showed how the quadratic model could be transformed to an equivalent model with a separable quadratic function making it much easier to solve with linear approximation approaches. Leung and Wong (2008 [69]) have developed a multivariate Sharpe ratio statistic to test the hypothesis of the equality of multiple Sharpe ratios.

Biglova et al. (2004 [20]) identified several other criteria for estimating portfolio theory risk that can be employed in LP models instead of the covariance risk measure of Markowitz (1952, 1997 [80, 81]). Among these include Gini's mean absolute difference as incorporated by Yitzhaki (1982 [158]) resulting in a LP for constructing efficient portfolios. In their linear optimization model, Konno (1990 [64]) and Konno and Yamazaki (1991 [65]) employed absolute deviation rather than covariance to measure the risk. They solved a problem with 224 stocks over 60 months on a real-time basis and found results similar to that of the mean-variance model but requiring much less computational effort. Speranza (1993 [136]) generalized this approach using a risk function that is a linear combination of two semi-absolute deviations of return from the mean.

Ogryczak (2000 [91]) formulated and solved a multi-objective LP consisting of one objective for each time period and showed the mean-variance approach of Markowitz (1952, 1997 [80, 81]), the absolute deviation approach of Konno and Yamazaki (1991 [65]), and the mini-max approach of Young (1998 [160]) to be special cases.

Young (1998 [160]) formulated an LP portfolio model for maximizing the minimum return to select a diversified portfolio based on historical returns data. He referred to the LP as a mini-max model because of its greater familiarity and this convention will be followed. The performance of the model was compared to other similar linear and nonlinear models and statistical analysis and simulation were employed to find that the mini-max approach outperformed the mean-variance approach with respect to mean square estimation error under the widely used log-normal distribution. He showed the mini-max modeling approach to be compatible with expected utility maximization and explored the incorporation of fixed transaction charges.

Cai et al. (2000 [29]) considered an objective of minimizing the expected absolute deviation of the future returns from their mean for several stocks and found that the problem could be solved analytically rather than solving a LP model. Similarly, Teo and Yang (2001 [147]) minimized the average of maximum individual risk over a number of time periods and the resulting optimization model was found to be solvable as a bi-objective piecewise LP problem. Benati (2003 [15]) replaced the covariance objective of Markowitz (1952, 1997 [80, 81]) with the worst conditional expectation resulting in a LP and developed an efficient algorithm for practical solutions to real-world sized problems. Ding (2006 [38]) considered LP models for maximizing the minimum returns but without the constraint for a minimum required average return for the portfolio as in Young (1998 [160]). For these simpler LP models he was able to develop optimal control policies for four cases of assumptions regarding evaluations (forecasts) for the potential returns. Gulpinar and Rustem (2007 [57]) proposed multiple alternative return and risk scenarios and developed a min-max algorithm to determine an optimal worst-case investment strategy.

Rockafellar and Uryasev (2000 [97]), Krokhmal et al. (2002 [67]), and Mansini et al. (2007 [79]) all focused upon minimizing Conditional Value-at-Risk ($CVaR$) and developed LP models, properties, and solution approaches for this setting. Schrage (2001 [129]) devoted a chapter to portfolio optimization featuring a LP model to maximize the minimum return and another to minimize expected downside risk. His wide-ranging treatment of this topic also included approximations for the covariance matrix, inclusion of transaction costs, and inclusion of taxes for the Markowitz (1952, 1997 [80, 81]) model as well as the Value-at-Risk (VaR) model and several deterministic equivalents of other stochastic optimization models. Alexander and Baptista (2002, 2004 [1, 2]) incorporated VaR and $CVaR$ as constraints in the Markowitz (1952, 1997 [80, 81]) model and found the $CVaR$ approach dominant for managing risk. Benati and Rizzi (2007 [16]) formulated an integer linear programming model with VaR replacing the covariance for the objective and developed properties for which polynomial time algorithms exist. Mansini et al. (2003a, b [77, 78]) provided a systematic overview, discussion of properties, and a computational comparison for LP solvable models for portfolio selection.

Kahneman and Tversky (1979 [62]) who have thoroughly examined under-weighting and over-weighting of probabilities as key issues, which may make insurance against losses attractive. These approaches for instance modeling began with Leland (1980 [68]) and Brennan and Solanki (1981 [26]) who examine maximizing expected utility subject to a budget constraint. But investor's preferences and probability belief may be difficult to ascertain and analyze thus Aliprantis et al. (2000 [3]) introduced a LP approach to minimize the cost of a portfolio subject to a minimal payoff. Katsikis (2007 [63]) further refined computational approaches for this model. Aliprantis et al. (2002 [4]) extended the LP model by taking advantage of the situation where portfolio dominance information is also available.

Gilboa and Schmeidler (1989 [56]) employed a set of multiple prior probability distribution to model situation where the decision maker has too little information to discern a single prior distribution and expressed investor preferences as a utility function over this set. Chateauneuf et al. (2005 [32]) developed theoretical underpinnings for a number of important applications of the multiple priors. Gajdos et al. (2004 [53]) introduced a partial order on the set of multiple priors based on a reference prior distribution within the set termed an anchor. They proceeded to show that a decision maker who is averse to information imprecisions tends to maximize the minimum expected utility with respect to a subset of the multiple priors. Garlappi et al. (2007 [54]) employed confidence intervals around estimated expected returns to reflect decision making under multiple prior and modeled ambiguity aversion in terms of minimization of a function over these priors.

Sawik (2008e [109]) constructed the three stage lexicographic approach and the corresponding mixed integer programming formulations for the multi-criteria portfolio optimization problem. The primary objective is to maximize expected portfolio return, then the minimization of risk probability of portfolio loss versus the maximization of amount of capital to be invested in portfolio is considered and finally, the minimization of number of stocks in optimal portfolio is achieved. Some additional examples of the portfolio multi-criteria mixed integer programming formulations with use the of VaR and $CVaR$ can be found in Sawik (2006a, 2006b, 2007a, 2007b, 2008a, 2008c, 2008d, 2009a, 2009b, 2009c, 2009d, 2009e, 2009f, 2009g, 2009h, 2010a, 2010b, 2010c, 2010f, 2011a, 2011b, 2012a, 2012b, 2012c [101, 102, 103, 104, 105, 107, 108, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 123, 124, 125, 126, 127, 128])(Presented in these publications multi-criteria portfolio optimization models are parts of this PhD dissertation.).

Polak et al. (2010 [94]) constructed mini-max portfolio model with linear programming approach. Employed risk measured as the worst-case return and a portfolio from maximizing returns subject to a risk threshold. They proceeded to show parametric analysis of the risk threshold connected their model to expected value along a continuum, revealing an efficient frontier segmenting investors by risk preferences.

Chen and Kwon (2010 [33]) developed a robust portfolio selection model for tracking a market index using a subset of its assets. The model is a binary program that seeks to maximize similarity between selected assets and the assets of the target index. Presented optimization model allows uncertainty in the objective function by using a computationally tractable robust framework that can control the conservativeness of the solution. This protects against worst-case realizations of potential estimation errors and other deviations.

Chen et al. (2011 [35]) developed tight bounds on the expected values of several risk measures. The basic settings was to find a portfolio that maximizes (respectively, minimizes) the expected utility (respectively, disutility) values in the midst of infinitely many possible ambiguous distributions of the investment returns fitting the given mean and variance estimation.

1.1.1 Definition of Multi-Objective Portfolio Optimization

A multi-objective optimization problem is formulated as follows:

Optimize (Maximize or Minimize) $F(x) = [f_1(x), ..., f_k(x)]$

Subject to $x \in X$

where $x = (x_1, ..., x_n)$ is the vector of decision variables and X is the set of feasible solutions. The objective function vector $F(x)$ which contains the values of k objectives maps the feasible set X into the set F (the feasible region in the objective space) which represents all possible values of the objective functions. The objective function may all be maximized, minimized or be in a mixed form. The usual process in multi-objective optimization is to find all non-dominated or Pareto optimal solutions of the problem, for instance, every solution which we cannot improve with one objective function without deteriorating another.

1.2 Definitions of Percentile Measures of Risk

Let $\alpha \in (0, 1)$ be the confidence level.

The percentile measures of risk, VaR and $CVaR$ can be defined as below:

- Value-at-Risk (VaR) at a $100\alpha\%$ confidence level is the targeted return of the portfolio such that for $100\alpha\%$ of outcomes, the return will not be lower than VaR. In other words, VaR is a decision variable based on the α-percentile of return, i.e., in $100(1-\alpha)\%$ of outcomes, the return may not attain VaR.

- Conditional Value-at-Risk ($CVaR$) at a $100\alpha\%$ confidence level is the expected return of the portfolio in the worst $100(1-\alpha)\%$ of the cases. Allowing $100(1-\alpha)\%$ of the outcomes not exceed VaR, and the mean value of these outcomes is represented by $CVaR$.

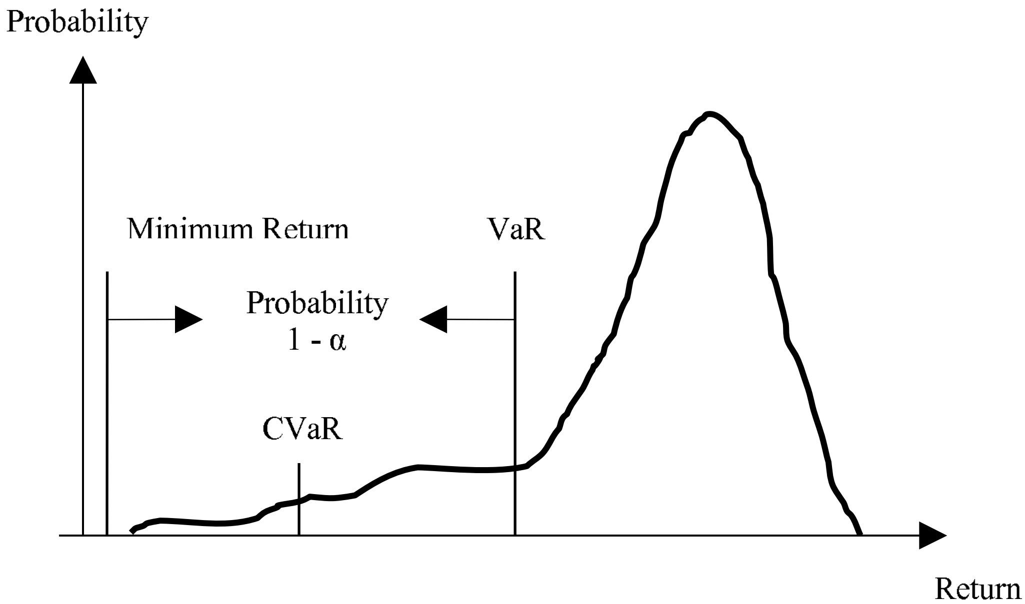

Figure 1.1: Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$)

Figure 1.1 illustrates Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) for a given portfolio and the confidence level α.

Value-at-Risk (VaR) represents the minimum return (maximal loss accepted by a decision maker; for instance $VaR = -1$ – minus one percent) associated with a specified confidence level of outcomes (i.e. the likelihood that a given portfolio's return will not be less than the amount defined as VaR).

However, VaR does not account for properties of the return distribution beyond the confidence level and hence does not explain the magnitude of the return when the VaR limit is exceeded.

On the other hand, $CVaR$ (Conditional Value-at-Risk) focuses on the tail of the return distribution, that is, on outcomes with the lowest return.

Since VaR and $CVaR$ measure different parts of the return distribution, VaR may be better for optimizing portfolios when good models for tails are not available, otherwise $CVaR$ may be preferred, e.g. (Rockafellar and Uryasev, 2000, 2002; Sarykalin et al., 2008; Uryasev, 2000 [97, 98, 100, 149]).

When using $CVaR$ to maximize worst-case return (minimizing maximal accepted portfolio loss), $CVaR$ is always less than VaR. On the other hand, VaR is a better choice to measure the risk of critical portfolio returns.

1.3 Percentile Measures of Risk in the Literature

Risk measures in portfolio optimization can be divided into two main categories Natarajan et al., 2009 [88]: moment based and quantile based. The roots of moment-based risk measures can be traced to classical economic utility theory, whereas quantile-based risk measures have arisen as a consequence of advances in the theory of stochastic dominance. Commonly used risk measures are mean-standard deviation, or equivalently mean-variance, Value-at-Risk (VaR), and finally Conditional Value-at-Risk ($CVaR$).

In the mean-variance context of Markowitz (1952, 1997 [80, 81]), the variance or standard deviation is adopted to measure the risk exposure of financial portfolios (see e.g. Bai et al., 2009a, 2009b; Leung and Wong, 2008; Sniedovich, 2007 [9, 10, 69, 135]). However, these measures fail to capture the downside risk. To circumvent this problem, many academics have proposed VaR (Holton, 2003; Jourion, 2006 [59, 61]) and the $CVaR$ (Alexander and Baptista, 2002, 2004; Sarykalin et al., 2008; Zhu and Fukushima, 2009 [1, 2, 100, 163]).

Generally risk management has received much attention from practitioners and regulators as well as academics in the last few years, with VaR emerging as one of the most popular tools (see Wong, 2011 [154]). Jourion (2006 [61]), Linsmeier and Pearson (2000 [73]), Alexander and Baptista (2002, 2004 [1, 2]), Hull (2003 [60]), and Chance (2004 [31]) note that VaR is widely used as a risk management tool by corporate treasurers, dealers, fund managers, financial institutions, and regulators (see Basel Committee on Banking Supervision 1996, 2003 [13, 14]).

In contrast, some researchers have extensively criticized the use of VaR as a measure of risk. For instance Artzner et al. (1999 [8]) pointed out that VaR is not a coherent measure of risk since it fails to hold the sub-additivity property. Moreover, VaR does not explain the magnitude of the loss when the VaR limit is exceeded. Furthermore, it is difficult to optimize when using calculated scenarios, and this leads to the use of an alternative measure, which is $CVaR$. Basak and Shapiro (2001 [12]) show that when an agent faces a VaR constraint at the initial date in a continuous-time model, the agent may select a larger exposure to risky securities than he or she would have chosen in its absence. Yiu (2004 [159]) shows that imposing a dynamic constraint in a continuous-time model leads an agent to select a smaller exposure to risky stocks than it would have been chosen in case of its absence. These reasons incline previously mentioned researchers to propose the use of $CVaR$ rather than VaR. Pflug (2000 [95]) proved that $CVaR$ is a consistent measure of risk for its sub-additivity and convexity properties. Uryasev (2000 [149]) presented a description of both: (a) an approach for minimizing $CVaR$ and (b) optimization problems with $CVaR$ constraints. Alexander and Baptista (2004 [2]) noticed that the presence of a VaR constraint will cause a slightly risk-averse agent to select a portfolio that has a smaller standard deviation than the one that would have been selected in its absence. However, there are also conditions under which the constraint causes a highly risk-averse agent to select a portfolio that has a larger standard deviation. $CVaR$ constraint is tighter than a VaR constraint when the $CVaR$ and VaR bounds coincide, these portfolio choice results are also true and to a greater extent if a $CVaR$ constraint is imposed. Therefore, a $CVaR$ constraint is more effective than a VaR constraint as a tool to control slightly risk-averse agents, but has a more perverse effect on highly risk-averse agents. However, this perverse result weakens or even disappears when a risk-free security is available, or the $CVaR$ bound is larger than the VaR bound. Moreover, if the $CVaR$ bound is set at a level so that $CVaR$ constraint has the same perverse effect on highly risk-averse agents as the VaR constraints. Then the $CVaR$ constraint will result in slightly risk-averse agents selecting portfolios with small standard deviations than those when a VaR constraint is imposed. If the $CVaR$ bound is set at an even larger level so that the $CVaR$ constraint decreases the standard deviations of the optimal portfolios of slightly risk-averse agents to select portfolios with smaller standard deviations than those when a VaR constraint is imposed. Hence, when the $CVaR$ bound is set between these two levels, a $CVaR$ constraint dominates a VaR constraint as a risk management tool.

The proposed multi-criteria portfolio approach allows aforementioned two percentile measures of risk in financial engineering: VaR and $CVaR$ to be applied for managing the risk of portfolio loss. The proposed mixed integer and linear programming models provide the decision maker with a simple tool for evaluating the relationship between expected and worst-case loss of portfolio return.

A risk measure can be linear program computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios.

VaR and $CVaR$ have been widely used in financial engineering in the field of portfolio management (e.g. Sarykalin et al., 2008 [100]). $CVaR$ is used in conjunction with VaR and is applied for estimating the risk with non-symmetric cost distributions. Uryasev (2000 [149]) and Rockafellar and Uryasev (2000, 2002 [97, 98]) introduced a new approach to select a portfolio with the reduced risk of high losses. The portfolio is optimized by calculating VaR and minimizing $CVaR$ simultaneously.

Polak et al. (2010 [94]) noticed that objectives such as minimizing variation or the popular VaR objective may be quite effective especially during periods of slow or moderate economic changes.

1.4 Mathematical Programming

The Mathematical programming methods of operations research for multi-criteria optimization are presented in this PhD dissertation. Not only for portfolio optimization of securities from stock exchanges, but also selected models for supporting services in medical care institutions, based on assignment problem.

The vast majority of the decision models are mathematical programming models (see e.g. Filipowicz, 1998; Ogryczak, 1997; Steuer, 1986; Toczyłowski, 2002; Zak, 2005; [46, 92, 138, 148, 161]).

The term "programming" was in use by 1940 to describe the planning or scheduling of activities within a large organization. "Programmers" found that they could represent the amount or level of each activity as a *variable* whose value was to be determined. Then they could mathematically describe the restrictions inherent in the planning or scheduling problem as a set of equations or inequalities involving the variables. A solution to all of these *constraints* would be considered an acceptable plan schedule (Dantzig, 1991; Fourer et al., 1990 [37, 47]).

Experience showed soon that it was hard to model a complex operation simply by specifying constraints. If there were too few constraints, many inferior solutions could satisfy them; if there were too many constraints, desirable solutions were ruled out, or in the worst case no solutions were possible. The success of programming ultimately depended on a key insight that provided a way around this difficulty. One could specify, in addition to the constraints, an *objective*: a function of the variables, such a cost or profit, that could be used to decide whether one solution was better than another. Then it didn't matter that many different solutions satisfied the constraints - it was sufficient to find one such solution that minimized or maximized the objective. The term *mathematical programming* came to be used to describe the minimization or maximization of an objective function of many variables, subject to constraints on the variables (Dantzig, 1991; Fourer et al., 2003 [37, 48]).

In the development and application of mathematical programming, one special case stands out; that in which all the costs, requirements and other quantities of interest are terms strictly proportional to the levels of the activities, or sums of such terms. In mathematical terminology, the objective could be a linear function, and the constraints are in such case linear equations and inequalities. Such a problem is called a *linear program*, and the process of setting up such a problem and solving it is called *linear programming*. Linear programming is particularly important because a wide variety of problems can be modeled as linear programs, and because there are fast and reliable methods for solving linear programs even with thousands of variables and constraints (Dantzig, 1991; Fourer and Gay, 2006 [37, 49]).

All useful methods for solving linear programs require a computer. Thus most of the study of linear programming has taken place since the late 1940's, when it became clear that computers would be available for scientific computing. The first successful computational method for linear programming, the simplex algorithm (Bertsimas and Tsitsiklis, 1997; Nemhauser and Wolsey, 1999 [18, 89]), was proposed at this time, and was the subject to increasingly effective implementations over the next decade. Coincidentally, the development of computers gave rise to a now much more familiar meaning for the term "programming" (Bisschop and Meeraus, 1982; Brooke at al., 1988 [21, 27]).

The assumption of linear programming also break down if some variables must take on whole (integer) number, or integral values. Then the problem is called *integer programming*, and in general becomes much harder. Nevertheless, a combination of faster computers and more sophisticated methods have large integer programs increasingly tractable in recent years (Fourer et al., 1990, 2003; Fourer and Gay, 2006; Fourer, 2007 [47, 48, 49, 50]).

Mathematical programming models presented in this PhD dissertation involve linear and integer variables, so for finding optimal solution of presented problems mixed integer programming was used.

1.4.1 Selected Mathematical Programming Methods of Multi-Objective Portfolio Optimization

Consider the following multi-objective problem (P):

maximize $z_1 = f_1(x)$

...

maximize $z_k = f_k(x)$

subject to $x \in X$,

where $X \subset \Re^n$ denotes the non-convex set of feasible solutions defined by a set of functional constraints, $x \geq 0$ and x_j integer $j \in J \subseteq 1, \ldots n$. It assumed that X is compact (closed and bounded) and non-empty. The integer variables can either be binary or take on general integer values. (P) is a multi-objective integer programming problem if all variables are integer. Otherwise (P) denotes a multi-objective mixed integer programming problem.

In linear multi-objective integer or mixed-integer problems, the functional constraints can be defined as $Ax \leq b$, and the objective functions $f_i(x) = c^i x$, $i = 1, \ldots, k$ where A is a $m \times n$ matrix, b is a m-dimensional column vector and c^i, $i = 1, \ldots, k$, are n-dimensional row vectors.

The concept of efficiency of non-dominance in multi-objective (mixed-)integer programming is defined as usually for multi-objective mathematical programming (Alves and Climaco, 2007; Ogryczak, 1997, Steuer, 1997; [5, 92, 139]).

A solution $\bar{x} \in X$ is $efficient$ for the problem (P) if and only if there is no $x \in X$ such that $f_i(x) \geq f_i(\bar{x})$ for all $i \in 1, \ldots, k$ and $f_i(x) > f_i(\bar{x})$ for at least one i.

A solution $\bar{x} \in X$ is $weakly - efficient$ for the problem (P) if and only if there is no $x \in X$ such that $f_i(x) > f_i(\bar{x})$ for all $i \in 1, \ldots, k$.

Let $\Re^k$ be the image of the feasible region X in the objective functions (criteria) space. A point $\bar{z} \in Z$ corresponding to a (weakly) $efficient$ solution $\bar{x} \in X$ is called (weakly) $non-dominated$.

Since the feasible region of (P) is non-convex, $unsupported\ non-dominated$ solutions may exist. A non-dominated point $\bar{z} \in Z$ is called unsupported if it is dominated by a convex combination (which does not belong to Z) of other non-dominated criteria points (belonging to Z) (Alves and Climaco, 2007 [5]).

1.4.2 Weighting and Lexicographic Approach

Mathematical programming approach deals with optimization problems of maximizing or minimizing a function of many variables subject to inequality and equality constraints and integrality (being, containing, or relating to one or more mathematical integers or relating to or concerned with mathematical integrals or integration) restrictions on some or all of the variables (Crescenzi and Kann, 2005; Merris, 2003; Nemhauser and Wolsey, 1999 [36, 86, 89]). In particular model equations consist of linear, integer and (representing binary choice) 0-1 variables. Therefore, the optimization models presented in this paper are defined as mixed integer or linear programming problems.

The lexicographic optimization generates efficient solutions that can be found by sequential optimization with elimination of the dominating functions. The weighted objective functions also generate various efficient solutions. It provides a complete parametrization of the efficient set for multi-criteria mixed integer programs.

An efficient solution to the multi-criteria portfolio optimization problem can be found by applying the weighting and lexicographic approach (Ehrgott, 2000; Sawik, 2007b, 2008e, 2009e, 2009g, 2010b; Steuer, 1986; Wiecek, 2007 [41, 104, 109, 114, 116, 119, 138, 152]).

The nondominated solution set of multi-objective mixed integer, linear or quadratic program models M (All optimization models presented in chapter 2.) can be partially determined by the parametrization on λ of the following weighted-sum program.

Model M_λ

Maximization or minimization $\sum_{l=1}^{m} \lambda_\iota f_\iota$

subject to some specific model constraints (As it is formulated in models presented in chapter 2.), where $\lambda_1 > \lambda_2 > ... > \lambda_m, \lambda_1 + \lambda_2 + ... + \lambda_m = 1$.

It is well known, however, that the nondominated solution set of a multi-objective mixed integer or linear or quadratic program such as M_λ cannot by fully determined even if the complete parametrization on λ is attempted (e.g., Steuer, 1986 [138]). To compute unsupported non-dominated solutions, some upper bounds on the objective functions should be added to M_λ (e.g., Alves and Climaco, 2007 [5]).

Considering the relative importance of the two or the three objective function (see optimization models presented in chapter 3) the multi-objective mixed integer or linear or quadratic program M can be replaced with M_ι, where $\iota \in 1, 2$ in case of two objective functions or $\iota \in 1, 2, 3$ in case of three objectives, that could be solved subsequently.

Model $M_\iota, \iota = 1, 2, 3$

Maximization or minimization f_ι

subject to some specific model equations (As it is formulated in models presented in chapter 2.) with additional constraints, in which upper or lower bounds are the optimal solution values of all objectives except the one with highest priority (f_l) - objective actually optimized:

$f_l = f_\iota^*; l < \iota : \iota > 1$, where f_ι^* is the optimal solution value to the mixed integer or linear or quadratic program $M_\iota, \iota = 1, 2$ (considering three objective lexicographic problems).

1.4.3 Reference Point Method

The reference point method (RPM) is a very effective technic for the multi-objective optimization problems.

The reference point method for LP and MIP programming is based on the Chebyshev metric (Alves and Climaco, 2007; Bowman, 1976; Skulimowski, 1996; Wiecek, 2007 [5, 23, 130, 152]).

Let us denote by $\|f(x) - \underline{\mathrm{f}}\|_\lambda$ the λ-weighted Chebyshev metric, i.e., $min_{1 \leq l \leq q}\{\lambda_l | f_l(x) - \underline{\mathrm{f}}|\}$, where $\lambda_l \geq 0 \; \forall l$, $\sum_{l=1}^{q} \lambda_l = 1$, and $\underline{\mathrm{f}}$ denotes a reference point of the criteria space. Considering $f(x) > \underline{\mathrm{f}}$ for all $x \in X$, it has been proven (Bowman, 1976 [23]) that the parametrization on λ of $min_{x \in X} \|f(x) - \underline{\mathrm{f}}\|_\lambda$ generates a non-dominated set.

The program $min_{x \in X} \|f(x) - \underline{\mathrm{f}}\|_\lambda$ may yield weakly non-dominated solutions, which can be avoided by considering the *augmented weighted Chebyshev* program:

Minimize $\delta + \gamma \sum_{l=1}^{q} f_l(x)$

subject to $\lambda_l(f_l(x) - \underline{\mathrm{f}}) \leq \delta$, $1 \leq l \leq q$

$x \in X$

$\lambda \geq 0,$

where γ is a small positive value. It has been proven (e.g. Steuer, 1986 [138]) that there always exists γ small enough that enable to reach all the non-dominated set for the finite-discrete and polyhedral feasible region cases (Alves and Climaco, 2007 [5]).

1.5 Selected Computational Methods for Mixed Integer Programming

There are three classical approaches for solving integer (IP) and mixed integer programs (MIP): *branch-and-bound*, *cutting plane* and *group theoretic*. Although all approaches are capable of solving integer and mixed integer programs, their degrees of success vary in software implementation. The cutting plane approach, when used as a stand-alone solver, has potential to solve IP programs of limited size, but may not work well in large-scale application. Similarly limited is the group theoretic approach, which has not been implemented as a stand-alone solver practice. However, the valid inequality cuts generated by both cutting plane and group theoretic approaches can be useful when combined with branch-and-bound to yield a powerful branch-and-cut approach (Chen et al., 2010 [34]).

The branch-and-bound had been the prevailing solution method until the emergence of the *branch-and-cut* in early 1990s. Branch-and-cut combined branch-and-bound with the generated cutting planes into a much more efficient "hybrid" approach. Similarly, the group cuts generated from the group theoretical approach have also been incorporated, but at a lesser degree of integration. As a whole, extracting the strengths of these two approaches and injecting them into the branch-and-bound may greatly increase the modern solution power for integer and mixed integer programs (Chen et al., 2010 [34]).

Branch-and-Bound is a general-purpose approach capable of solving pure IP, mixed IP, and binary IP problems. Theoretical, any pure IP problem with *finite* bounds on integer variables can be solved by enumerating all possible combinations of integer values and determining a combination (solution) that satisfies all constraints and yields the maximal (minimal) objective value - hence the name of *complete enumeration*. Unfortunately, the number of all possible combinations is prohibitively large to be evaluated even for a small problem. A problem of n integer variables with m values each has a total of m^n possible combinations (feasible and infeasible solutions). Therefore, complete enumeration is theoretically simple but practically intractable (Chen et al., 2010 [34]).

As a better alternative, *implicit enumeration* applies an intelligent enumeration scheme that can cover all possible solution by explicitly evaluating only a small number of solutions while ignoring (or implicitly enumerating) a large number of *inferior* solutions. One such strategy is called *divide and conquer*. Basically, this strategy *divides* the given problem into a series of easier to solve subproblems that are systematically generated and solved (or *conquered*). The solutions of these generated subproblems are then put together to solve the original problem (Chen et al., 2010 [34]).

1.5.1 Branch-and-Bound

Branch-and-bound can be viewed as a divide-and-conquer approach to solving the IP problem, in which a branching process for dividing and a bounding process for conquering are used. Throughout the algorithm, a series of LP subproblems are systematically generated and solved. Then upper and lower bounds are progressively tightened on the objective value of the original IP problem (Chen et al., 2010 [34]).

A typical way of represent such a process is via a *branch-and-bound (B&B) tree*, which is a specialized enumeration tree for keeping track of how LP subproblems are generated and solved. The B&B tree by convention drawn upside down with its root node at the top. The root node that represents the linear programming (LP) relaxation of the original IP problem is solved. If the LP optimum solution satisfies the integer requirement, the IP problem is solved. Otherwise, the LP objective value becomes the initial upper bound on the IP optimal objective value and the root node is partitioned into two successor nodes (subproblems) by two branches. These branches are valid cuts in term of simple inequality constraints that have the following properties:

- they cut off the current non-integer LP optimum point and other fractional region,
- the two successor nodes are mutually exclusive and their union contains the same integer feasible region as that of their predecessor (i.e. no integer points are eliminated).

The solution of an LP relaxation on a node provides information about whether a further branching from this node is needed (or whether the node can be *pruned*[1]), and a better lower bound (for maximization problem) on the objective of the original IP problem (Chen et al., 2010 [34]).

[1]In some texts about B&B method, the term *pruned* may be replaced by *fathomed*, to indicate that no further exploration beyond that point is necessary.

There are three cases indicating that a node can be pruned:

1. the subproblem has no feasible LP solution

2. the subproblem has an integer optimum solution

3. the upper bound of the subproblem optimum is less than or equal to the lower bound of the original problem.

These three cases are, respectively, referred to as *pruned by infeasibility*, *pruned by optimality*, *pruned by bound*. If a node is pruned by optimality, its optimum solution can be used to increase the lower bound on the objective value of the original IP problem (Chen et al., 2010 [34]).

Whenever an integer solution to a subproblem is obtained, it is a *candidate* optimum to the original IP problem. In the solution process of B&B, the best integer solution found so far is continuously updated. Such a solution is called an *incumbent solution* (Chen et al., 2010 [34]).

1.5.2 Cutting Plane

In geometry, an equation in two variables is called a *plane* and an equation in n variables a *hyperplane*, speaking strictly. However, both in practice are often referred to as a plane, regardless of the number of variables. An inequality constraint in n variables is called a *half-space*, not a hyper-plane. But an inequality constraint can always be converted to an equation by adding or subtracting a nonnegative slack variable. The term *cutting plane* is often used for an equality or inequality constraint that can cut off a fractional part of an LP feasible region, without excluding any *integer* feasible solution. In the cutting plane approach, one or more such cutting planes are added to the current LP simplex tableau, which in turn are resolved for a new LP optimum. This process is repeated until the prescribed integer requirements are satisfied (Chen et al., 2010 [34]).

The cutting plane approach often takes a large number of cuts to reach an integer solution even for a small or moderate sized IP problem, although it can be shown that the fractional cutting plane method is ensured to converge to an IP optimum after a finite number of cuts (Chen et al., 2010 [34]).

1.5.3 Branch-and-Cut

Since the development of the branch-and-bound (B&B), cutting plane, and group theoretical approaches in the 1960s, progress on methods for solving large-scale IP and MIP problems was very limited for two decades. Then in the mid-1980s, a novel solution approach known as *branch-and-cut* (B&C) was introduced, which marked a breakthrough milestone in the power of MIP solution algorithms. This approach and its variations, coupled with the advances in modeling techniques, preprocessing techniques, LP software, and computer hardware, make the solution of large-scale MIP problems possible. As of today, the solution power has leapt from solving problems with up to one hundred integer variables in the early 1980s to solving problems with thousands of integer variables, and even in many instances with millions of 0-1 variables (Chen et al., 2010 [34]).

1.5.4 Branch-and-Price

Branch-and-bound is first generalized to include generation of *columns* by solving pricing problems, hence the name *branch-and-price*, and yet another generalization includes generation of columns and rows, hence the name *branch-and-price-and-cut*. Basically, all these generalization solve a sequence of LP relaxations of a given IP. Branch-and-cut tightens the LP relaxation (or polyhedra) by adding cuts or constraints (rows). Branch-and-price tightens the LP relaxations by generating a subset of profitable columns associated with variables to join the current basis. These columns are generated iteratively by solving subproblems or *pricing problems* (Chen et al., 2010; Toczyĺowski, 2002 [34, 148]).

1.6 Commercial Software Solutions and Solvers for Mathematical Programming

After a linear program (LP) or mixed integer program (MIP) is formulated, some computer software package (e.g. the CPLEX solver) is typically used to solve the problem. CPLEX is from ILOG, and IBM company. Hence, an input mechanism is needed to translate the mathematical/algebraic description of the problem into a format that the software recognizes. Such input mechanisms are often offered to as modeling tools. Fourer et al. (1990, 2003, 2006, 2007 [47, 48, 49, 50]) call this the problem of translation from "modeler's form" to "algorithm's form", the latter referring to the simplex algorithm and the simplex-based branch-and-bound and branch-and-cut methods, found in commercial solvers. Common modeling tools for LP or MIP problems were developed chronologically, and they fall into three categories:

1. MPS format files,
2. LP-format files,
3. Algebraic modeling languages.

1.6.1 AMPL Programming Language

Practical mathematical programming is seldom as simple as running some algorithmic method on a computer and printing the optimal solution. The full sequence of events is more like this:

- Formulate the model - the abstract system of variables, objectives, and constraints that represent the general form of the problem to be solved.
- Collect data that define one or more specific problem instances.
- Generate a specific objective function and constraint equations from the model and data.
- Solve the problem - run a program to apply an algorithm that finds optimal values of the variables.
- Analyze the results.
- Refine the model and data as necessary, and repeat.

AMPL belongs to filed of algebraic modeling languages for mathematical programming. This language is notable for the similarity of its arithmetic expressions to customary algebraic notations, and for the generality of its set and subscripting expressions. AMPL language extends algebraic notation to express common mathematical programming structures.

The computational experiments for this PhD dissertation have been performed mainly using AMPL programming language (Fourer et al., 1990, 2003 [47, 48]) and the CPLEX v.11 solvers (with the default settings) on a laptop with Intel Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM.

1.6.2 LINGO Programming Language

LINGO is a Fortran-based optimization tool designed by LINDO Systems, Inc., first offered in 1988. A unique features of LINGO is that all solvers (linear, integer, nonlinear, quadratic, etc.) are integrated and directly linked to its modeling environment. When a model is run, LINGO will automatically pass the problem to the appropriate solver. Hence, LINGO is capable of solving a wide variety of optimization problems, including linear programming, integer programming (binary, pure, and mixed), and nonlinear programming problems (LINDO Sys., 2004 [72]).

1.6.3 MPL Programming Language

MPL [85], which stands for Mathematical Programming Language, is a product of Maximal Software, Inc. MPL is another algebraic modeling language, such as AMPL and LINGO. Some notable features of building optimization models in MPL are:

- MPL can dynamically store models of any size like LINGO; the only limitation is how much memory is available on the machine
- Variables and constraints can be written on both sides of a constraint - the so-called free format input of constraints, which means no conversion to standard form is required of the modeler
- Expansion of similarly structured constraints; a single line enables to express multiple constraints of identical form, such as monthly inventory balance someone want to repeated for each month in a planning horizon; for instance.
- Extensive flexibility when working with subsets of indexes, functions of indexes, and multidimensional index sets.

1.6.4 MATLAB Optimization Tools for mathematical programming

Optimization Toolbox in MATLAB provides widely used algorithms for standard and large-scale optimization. These algorithms solve constrained and unconstrained continuous problems. The toolbox includes functions for linear programming, quadratic programming, nonlinear optimization, nonlinear least squares, systems of nonlinear equations, and multi objective optimization. This toolbox can be used to find optimal solutions, perform tradeoff analysis, balance multiple design alternatives, and incorporate optimization methods into algorithms and models. Optimization Toolbox includes the most widely used methods for performing minimization and maximization. The toolbox implements both standard and large-scale algorithms, enabling you to solve problems by exploiting their sparsity or structure [84].

Optimization Toolbox includes three algorithms used to solve linear programming problems: interior point, active-set, and simplex. The interior point algorithm is based on a primal-dual predictor-corrector algorithm used for solving linear programming problems. Interior point is especially useful for large-scale problems that have structure or can be defined using sparse matrices. The active-set algorithm minimizes the objective at each iteration over the active set (a subset of the constraints that are locally active) until it reaches a solution. The simplex algorithm is a systematic procedure for generating and testing candidate vertex solutions to a linear program. The simplex algorithm is the most widely used algorithm for linear programming [84].

This MATLAB toolbox implements two algorithms for solving quadratic programming problems: large-scale and medium-scale. The large-scale algorithm switches between the trust-region reflective algorithm and the preconditioned conjugate gradient algorithm. The medium-scale algorithm uses the active-set algorithm. The trust-region reflective algorithm is used for bound constrained problems. The preconditioned conjugate gradient algorithm is used for problems subject to equality constraints. The active-set algorithm is used for problems that have inequality constraints or bounds and equalities [84].

Multi-objective optimization is concerned with the minimization of multiple objective functions that are subject to a set of constraints. MATLAB Optimization Tools provides functions for solving two formulations of multi-objective optimization problems: goal attainment and mini-max. The goal attainment problem involves reducing the value of a linear or nonlinear vector function to attain the goal values given in a goal vector. The relative importance of the goals is indicated using a weight vector. The goal attainment problem may also be subject to linear and nonlinear constraints. The minimax problem involves minimizing the worst-case value of a set of multivariate functions, possibly subject to linear and nonlinear constraints. Optimization Toolbox transforms both types of multi-objective problems into standard constrained optimization problems and then solves them using an active-set approach. Global Optimization Toolbox provides a multi-objective genetic algorithm solver for calculating multi-objective Pareto fronts [84].

1.7 The Input Data Sets

The input data used for computational experiments consist of four sets of 500, 1000, 2000 and 3000 historical daily quotations from the Warsaw Stock Exchange.

First data set includes 240 securities (i.e. stocks) with daily percentage price returns of each stock for period of 500 days from January 31st, 2007 to February 2nd 2009.

Second input data set for numerical experiments consist of 120 stocks with 1000 days from March 4th, 2005 to February 2nd, 2009.

Third set of historical quotations includes 127 securities with 2000 historical returns from time period since February 11th, 1999 till February 2nd, 2007.

Finally, fourth data set incorporates percentage price returns of 46 securities quoted each day in the Warsaw Stock Exchange during March 10th, 1997 to February 2nd, 2009, altogether 3000 days.

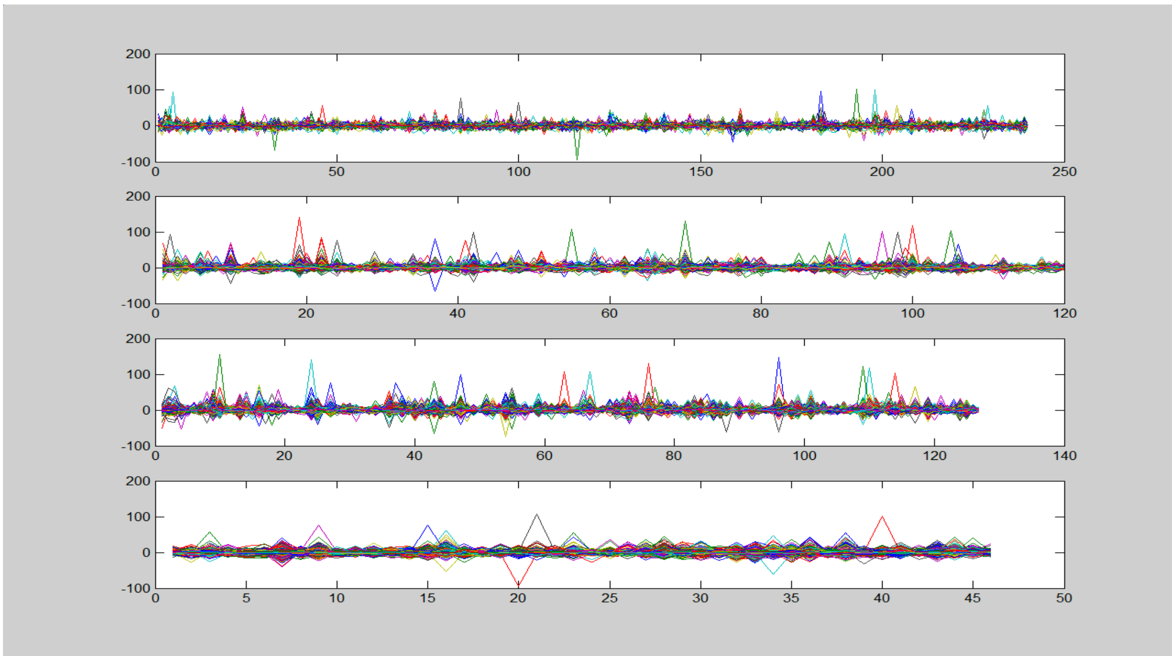

Figure 1.2: Historical daily returns of stocks from 500, 1000, 2000, 3000 data sets

Figure 1.2 illustrates historical daily returns of stocks from input data sets consist of 500, 1000, 2000, and 3000 historical daily quotations of 240, 120, 127 and 46 stocks.

Histograms presented below in figures from 1.3 to 1.15 show the statistical analysis of input data sets. For each histogram a selected data set has been divided into 250 days. For each period of 250 days mean value, standard deviation and Chi-square test ratio has been calculated. Normal distribution is presented by a theoretical line and confidence level is illustrated by the points above and under the line.

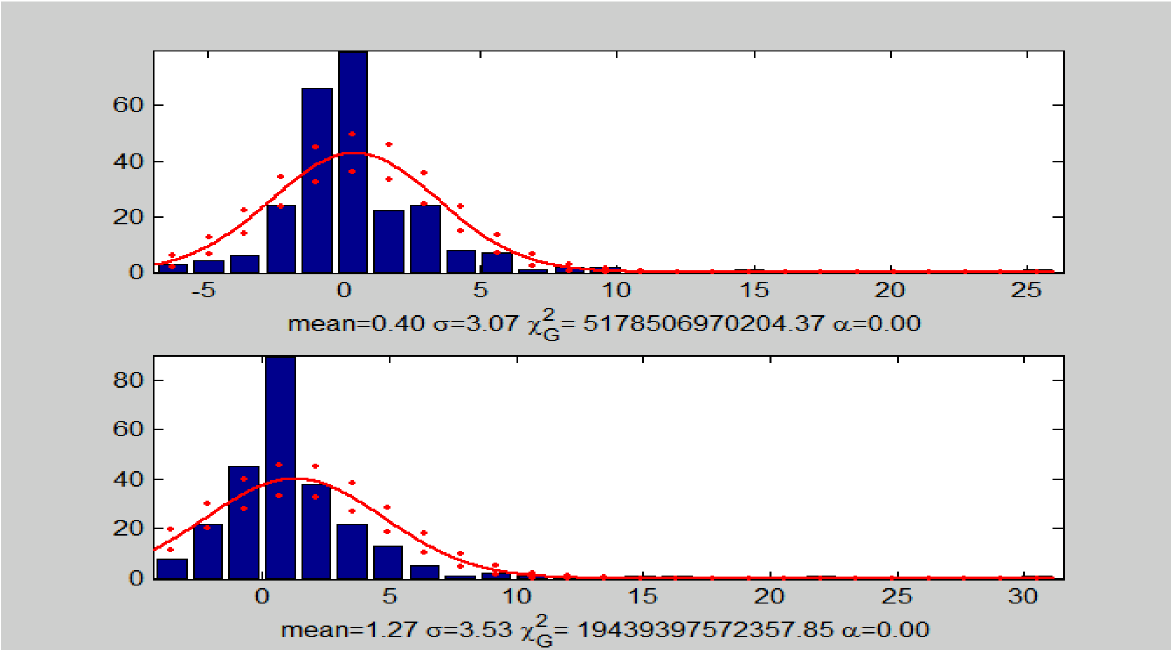

Figure 1.3: Histograms of input data set of 500 days with daily returns of 240 stocks. Periods of time form day 1 to 250 and from day 251 to 500

Figure 1.3 presents two histograms of input data with 500 days and 240 stock from January 31st, 2007 to February 2nd 2009 separate into two parts of 250 days in each histogram.

Figure 1.4 shows histograms for time period of 1000 days data set from first to 250th day and from 251st to 500th day.

Figure 1.5 illustrates two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 120 stocks with historic quotations from 501st and 751st day.

Figure 1.6 presents two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 127 stocks with historic quotations from first and 251st day.

Figure 1.7 shows two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 127 stocks with historic quotations from 501st and 751st day.

Figure 1.8 illustrates two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 127 stocks with historic quotations from 1001st and 1251st day.

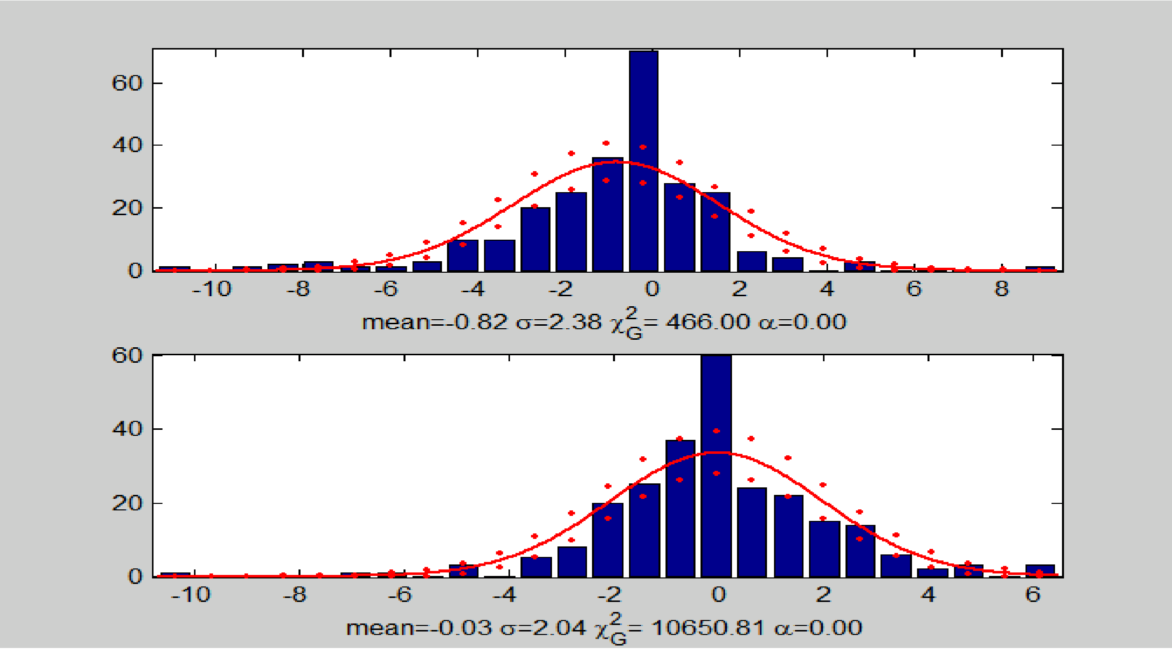

Figure 1.4: Histograms of input data set of 1000 days with daily returns of 120 stocks. Periods of time form day 1 to 250 and from day 251 to 500

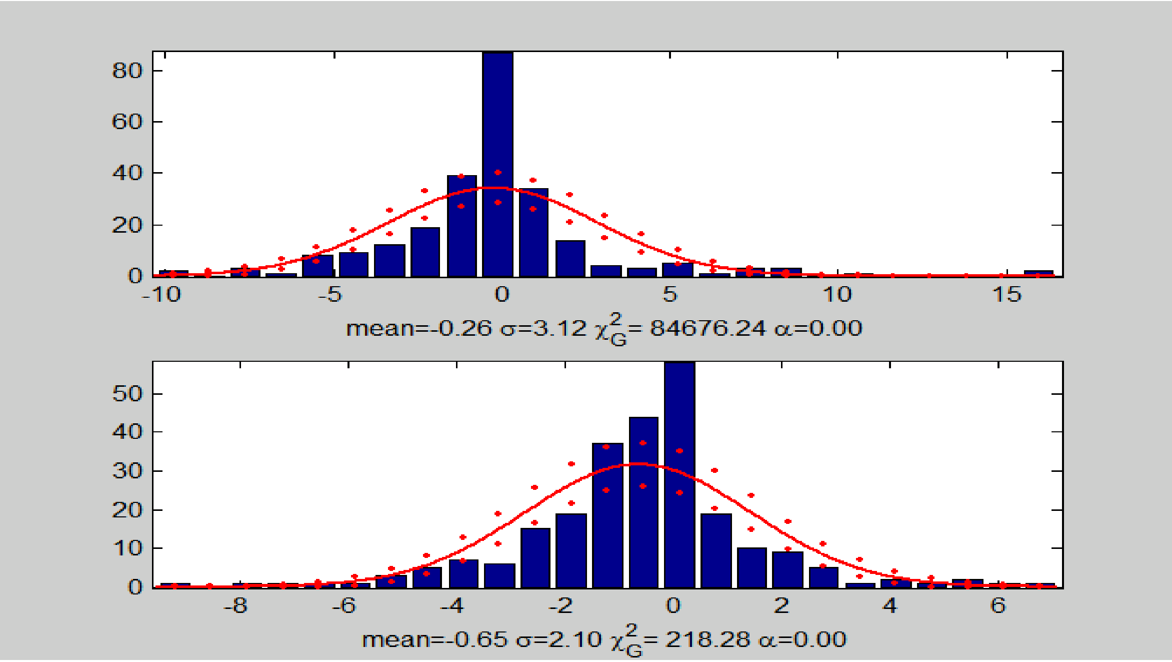

Figure 1.5: Histograms of input data set of 1000 days with daily returns of 120 stocks. Periods of time form day 501 to 750 and from day 751 to 1000

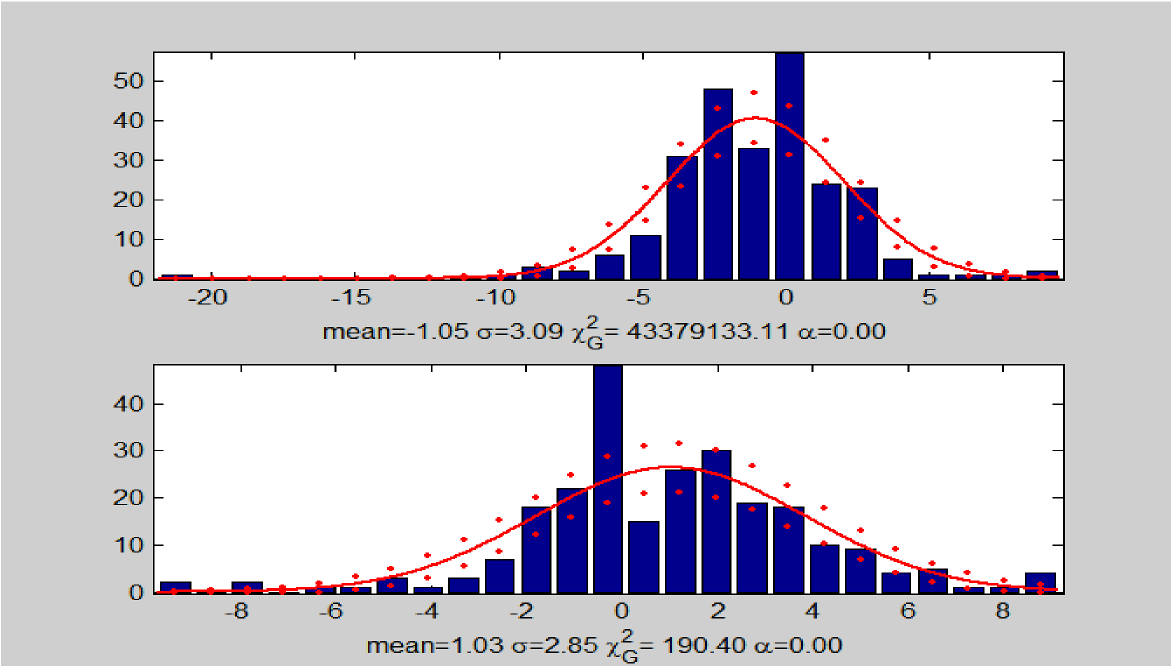

Figure 1.6: Histograms of input data set of 2000 days with daily returns of 127 stocks. Periods of time form day 1 to 250 and from day 251 to 500

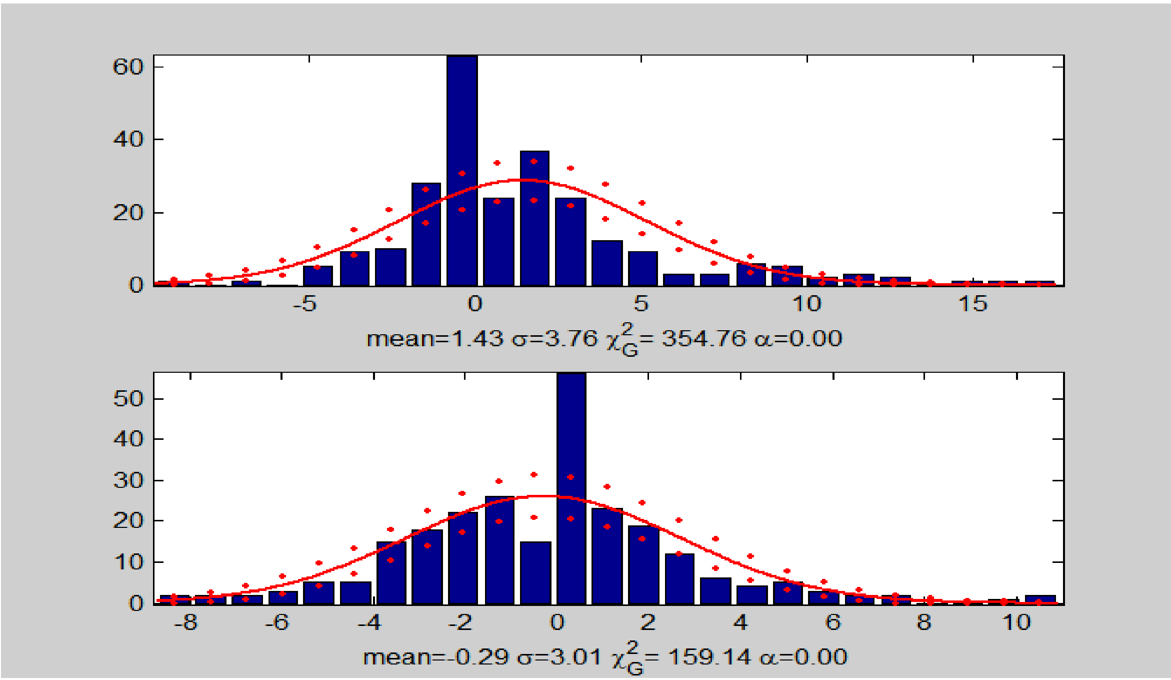

Figure 1.7: Histograms of input data set of 2000 days with daily returns of 127 stocks. Periods of time form day 501 to 750 and from day 751 to 1000

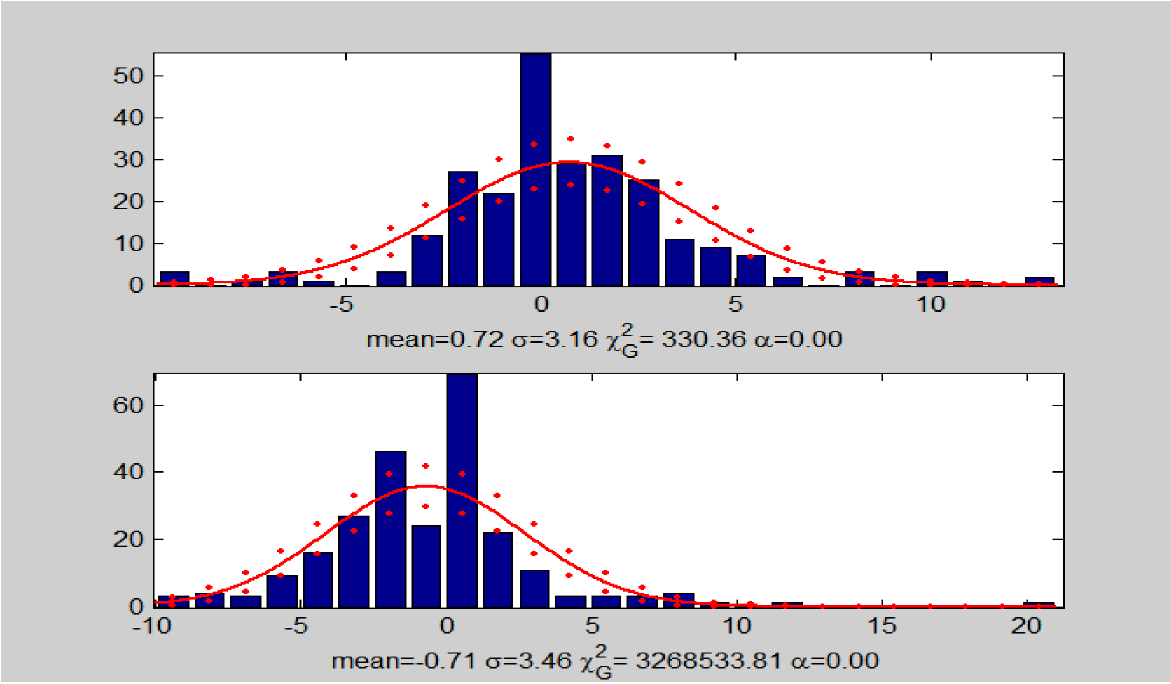

Figure 1.8: Histograms of input data set of 2000 days with daily returns of 127 stocks. Periods of time form day 1001 to 1250 and from day 1251 to 1500

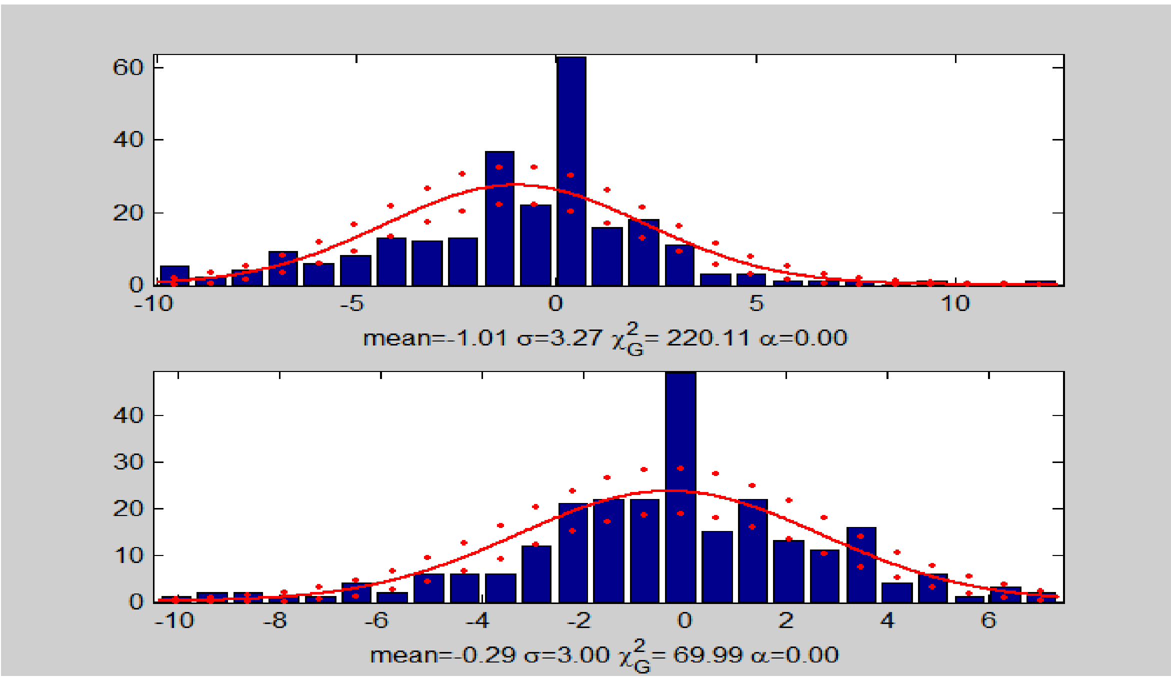

Figure 1.9: Histograms of input data set of 2000 days with daily returns of 127 stocks. Periods of time form day 1501 to 1750 and from day 1751 to 2000

Figure 1.9 presents two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 127 stocks with historic quotations from 1501st and 1751st day.

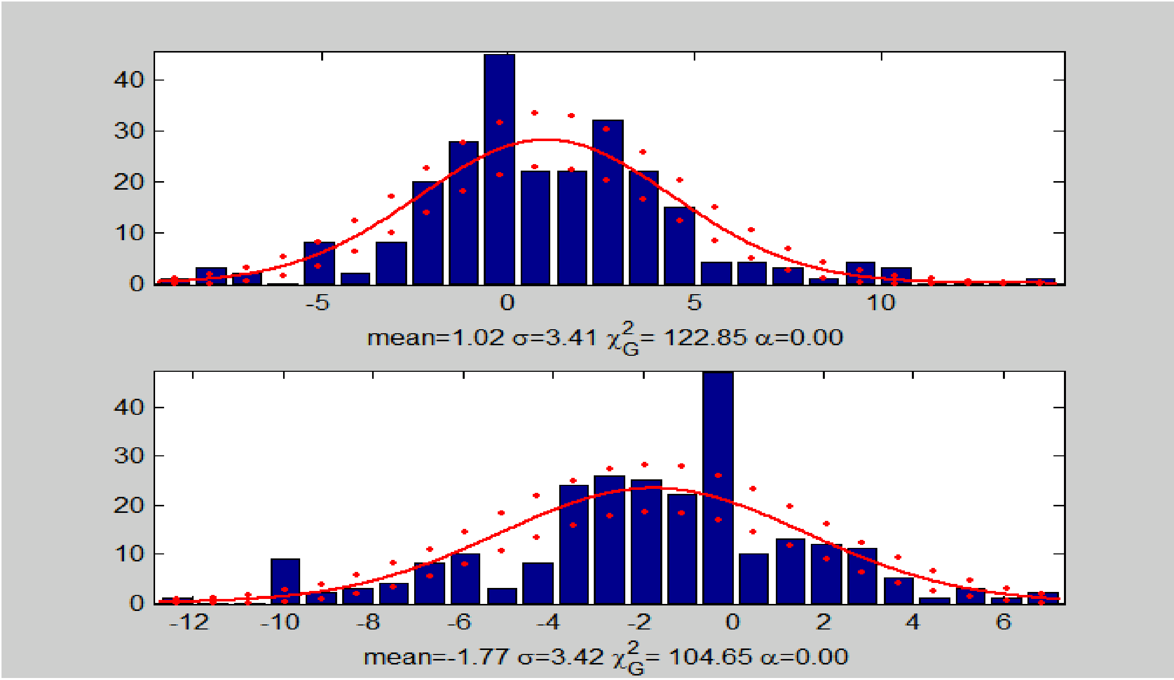

Figure 1.10: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 1 to 250 and from day 251 to 500

Figure 1.10 shows two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks with historic quotations from first and 251st day.

Figure 1.11 illustrates two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks with historic quotations from 501st and 751st day.

Figure 1.12 presents two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks with historic quotations from 1001st and 1251st day.

Figure 1.13 shows two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks with historic quotations from 1501st and 1751st day.

Figure 1.14 illustrates two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks

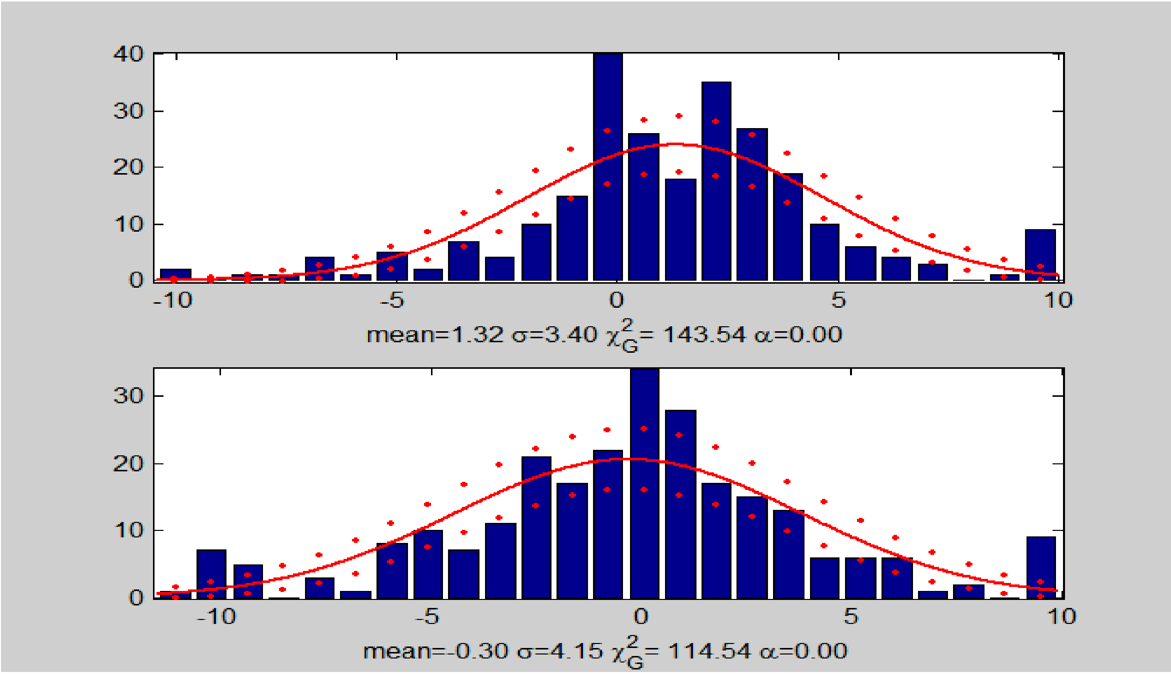

Figure 1.11: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 501 to 750 and from day 751 to 1000

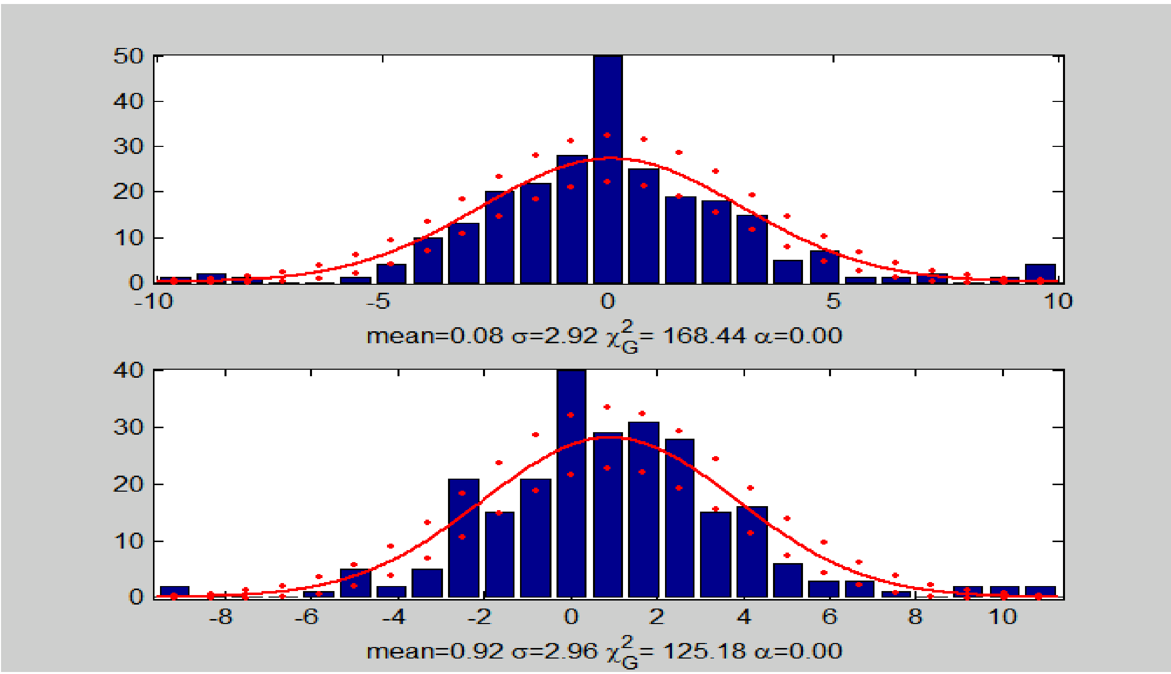

Figure 1.12: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 1001 to 1250 and from day 1251 to 1500

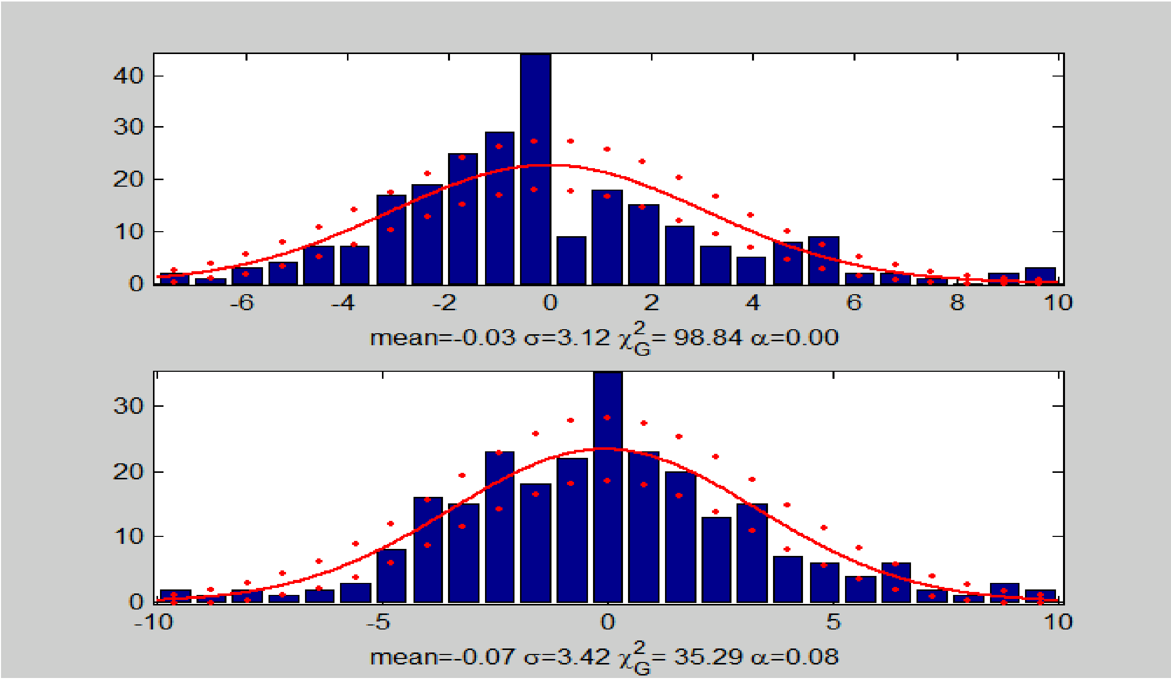

Figure 1.13: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 1501 to 1750 and from day 1751 to 2000

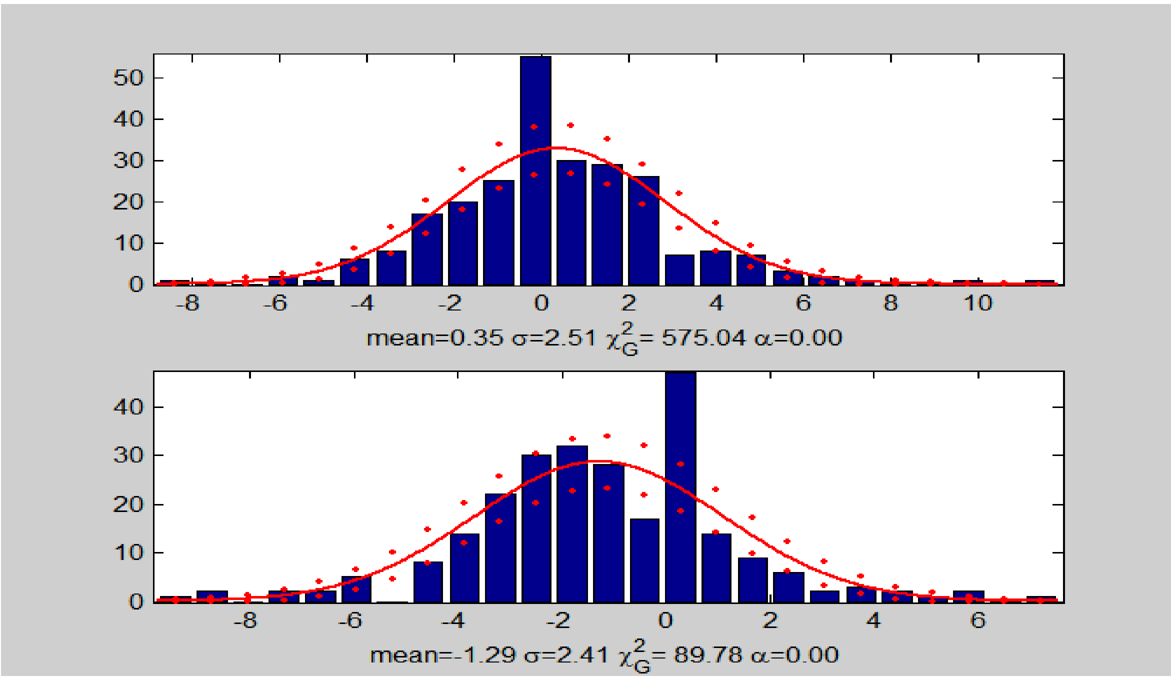

Figure 1.14: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 2001 to 2250 and from day 2251 to 2500

with historic quotations from 2001st and 2251st day.

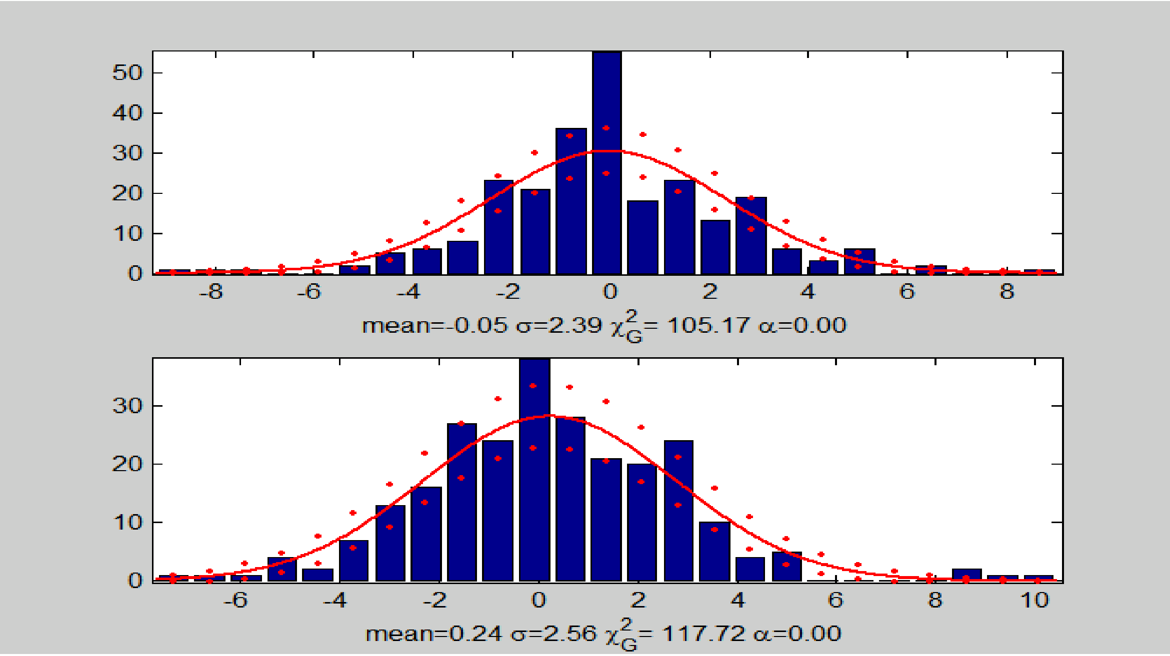

Figure 1.15: Histograms of input data set of 3000 days with daily returns of 46 stocks. Periods of time form day 2501 to 2750 and from day 2751 to 3000

Figure 1.15 presents two histogram for 250 days, each - with calculated mean values, standard deviations and Chi-squared ratios for data set consist of 46 stocks with historic quotations from 2501st and 2751st day.

Chapter 2

Weighting Approach to Multi-Objective Portfolio Optimization

The non-dominated solution set of multi-objective mixed integer, linear or quadratic program models M (All optimization models presented in this chapter.) can be partially determined by the parametrization on λ of the following weighted-sum program.

Model M_λ

Maximization or minimization $\sum_{l=1}^{m} \lambda_\iota f_\iota$

subject to some specific model constraints (As it is formulated in models presented in this chapter.), where $\lambda_1 > \lambda_2 > ... > \lambda_m, \lambda_1 + \lambda_2 + ... + \lambda_m = 1$.

It is well known, however, that the nondominated solution set of a multi-objective mixed integer or linear or quadratic program such as M_λ cannot by fully determined even if the complete parametrization on λ is attempted (e.g., Steuer, 1986 [138]). To compute unsupported non-dominated solutions, some upper bounds on the objective functions should be added to M_λ (e.g., Alves and Climaco, 2007 [5]).

2.1 Bi-Objective Portfolio Models with Objectives of Portfolio Return and Risk

This section includes bi-objective portfolio models. Weighting approach for objective functions has been implemented. The first objective defines risk of portfolio venture, this objective minimizes risk subject to specific constraints. The second objective function maximizes portfolio expected return. In the first subsection of this chapter presented bi-objective portfolio model is constructed with implementation of conditional value-at-risk as a main risk measure. The portfolio model presented in the second subsection has value-at-risk as a basic measure of risk. Third subsection of this chapter includes modified classical Markowitz portfolio bi-objective model, which is added for results comparison between presented portfolio approaches.

Table 2.1: Notations for mathematical models M1, M2, M3, M4, M5, M6

		Indices
i	=	historical time period, $i \in I = \{1, \ldots, m\}$ (i.e. day, week, month, etc.)
j	=	security, $j \in J = \{1, \ldots, n\}$
		Input parameters
α	=	input parameter in selected problems - confidence level. The mathematical models, where α is the input parameter are M1, M4.
$\beta_1, \beta_2, \beta_3, \lambda$	=	weights in the objective functions
$cov(r_i, r_j)$	=	matrix of covariance - the input parameter in the mathematical models: M3, M6.
p_i	=	probability assigned to the occurrence of past realization i
r_{ij}	=	observed return of jth stock in ith time period
r^{Min}	=	minimum return observed in the market
VaR	=	return Value-at-Risk. The mathematical models, where VaR is the input parameter are M2, M5.

2.1.1 Conditional Value-at-Risk Bi-Criteria Portfolio Model

The proposed model **M1** provides a decision maker with a tool for evaluating the relationship between expected and worst-case returns. The portfolio problem presented (Sawik, 2010f [123]) below provides flexibility in how a decision maker wants to balance his/hers risk tolerance with the expected portfolio returns. The bi-criteria weighted-sum portfolio problem consists of two objective functions (2.1). The first objective is to maximize Conditional Value-at-Risk ($CVaR$) and the second objective is to maximize portfolio expected return.

Chapter 3

Lexicographic Approach to Multi-Objective Portfolio Optimization

This chapter includes bi- and triple-objective portfolio models. Lexicographic approach for objective functions has been implemented. The first objective defines risk of portfolio venture, this objective minimizes risk subject to specific constraints, including constraints with following objectives placed as upper and lower bounds values. The second objective function, which is maximized has been defined as portfolio expected return. The third objective function is the number of securities in optimal portfolio. In the first subsection of this chapter presented multi-objective portfolio models are constructed with implementation of conditional value-at-risk as a main risk measure. The portfolio models presented in the second subsection have value-at-risk as a basic measure of risk. The third subsection of this chapter includes modified classical Markowitz portfolio multi-objective models, which are added for results comparison between presented portfolio approaches.

Then one criterion is maximized or minimized in objective function, the following are upper or lower bounds in optimization models constraints.

Considering the relative importance of the two or the three objective function (see optimization models presented in this chapter) the multi-objective mixed integer or linear or quadratic program M can be replaced with M_ι, where $\iota \in 1, 2$ in case of two objective functions or $\iota \in 1, 2, 3$ in case of three objectives, that could be solved subsequently.

Model $M_\iota, \iota = 1, 2, 3$

Chapter 4

Reference Point Approach to Multi-Objective Portfolio Optimization

This chapter includes bi- and triple-objective portfolio models. Reference Point Method (in which the relative importance of the objective functions is weighted in the constraint set) has been implemented as an approach for multi-criteria objective functions. The first objective defines risk of portfolio venture, this objective minimizes risk subject to specific constraints. The second objective function maximizes portfolio expected return. The third objective function is the number of securities in an optimal portfolio. In the first subsection of this chapter a presented multi-objective portfolio model is constructed with implementation of Conditional Value-at-Risk ($CVaR$) as a main risk measure. The portfolio model presented in the second subsection has Value-at-Risk (VaR) as a basic measure of risk. Third subsection of this chapter includes modified classical Markowitz portfolio multi-objective model, which is added for the results comparison between presented portfolio approaches.

4.1 Bi-Objective Portfolio Models

4.1.1 Conditional Value-at-Risk Portfolio Model

In the objective function (4.1) of the bi-criteria $CVaR$ portfolio optimization model **M16** with the augmented weighted Chebyshev metric is presented. The portfolio criteria aims are Conditional Value-at-Risk and the expected return in the portfolio.

$$x_j \geq 0; j \in N \tag{4.27}$$

Constraint (4.27) (see equation (2.5)) defines continuous variable x_j - percentage of wealth that is allocated to security (asset) j. This formula prevents short-selling.

$$\delta \geq 0 \tag{4.28}$$

Equation (4.28) (see (4.9)) is a non-negativity condition for deviation from the reference solution.

4.2 Triple-Objective Portfolio Models

This section includes triple-objective portfolio models. Chebyshev program (in which the relative importance of the objective functions is weighted in the constraint set) for objective functions has been implemented. The first objective defines risk of portfolio venture, the second objective function, which is maximized has been defined as portfolio expected return, finally the third objective has been defined as the number of stocks in portfolio. In the first subsection of this chapter a presented triple-objective portfolio model is constructed with implementation of Conditional Value-at-Risk $CVaR$ ($CVaR$) as a main risk measure. The portfolio model presented in the second subsection has Value-at-Risk (VaR) as a basic measure of risk. The third subsection of this chapter includes modified classical Markowitz portfolio triple-objective model, which is added for the results comparison between presented portfolio approaches.

4.2.1 Conditional Value-at-Risk Portfolio Model

In the objective function (4.29) of the triple-criteria $CVaR$ portfolio optimization model **M19** with the augmented weighted Chebyshev metric is presented. The portfolio criteria aims consist of Conditional Value-at-Risk ($CVaR$), expected return in portfolio and the number of assets[2]) in the optimal portfolio.

Model M19.

Minimize

[2]Maximization or minimization according to a decision maker preferences.

Chapter 5

Selected Multi-Period Portfolio Models

This chapter includes...

5.0.4 Problem Formulation

Suppose that n securities are available in the market with historical quotations in t intervals (investment periods), each of h multi-period intervals, in total m samples.

Table 5.1: Notations for mathematical model M22

		Indices
i	=	historical time period, $i \in I = \{1, \ldots, m\}$ (i.e. day)
j	=	security, $j \in J = \{1, \ldots, n\}$
k	=	historical multi-period interval $k \in K = \{1, \ldots, t\}$ (i.e. year, quarter or month, etc)
		Input parameters
h	=	number of historical quotations in each multi-period interval
p_i	=	probability assigned to the occurrence of past realization i
r_{ij}	=	observed return of jth stock in ith time period
r^{Min}	=	minimum return observed in the market
VaR	=	return Value-at-Risk
α	=	confidence level

Chapter 6

Alternative Portfolio Formulations

This chapter includes alternative portfolio formulations. The explanations that some computational results obtained for selected models can be used for comparison with other models presented in this dissertation is also included in this chapter, together with mathematical proof.

Table 6.1: Notations for mathematical models M24, M25, M26, M27

		Indices
i	=	historical time period, $i \in I = \{1, \ldots, m\}$ (i.e. day, week, month, etc.)
j	=	security, $j \in J = \{1, \ldots, n\}$
		Input parameters
α	=	input parameter in selected problems - confidence level. The mathematical models, where α is the input parameter are M24.
$\beta_1, \beta_2, \beta_3, \beta_4, \lambda$	=	weights in the objective functions
p_i	=	probability assigned to the occurrence of past realization i
r_{ij}	=	observed return of jth stock in ith time period
r^{Min}	=	minimum return observed in the market
VaR	=	return Value-at-Risk

jective function, given $V = VaR$ with fixed α parameter.

The computational results obtained for model **M24** can be used for comparison with model **M3** (modified Markowitz portfolio).

Similarly, the results obtained for model **M1** can be used for comparison with model **M2** for the same values of α parameter.

Model **M1** could be also used for finding optimal value of VaR for different value of confidence level α.

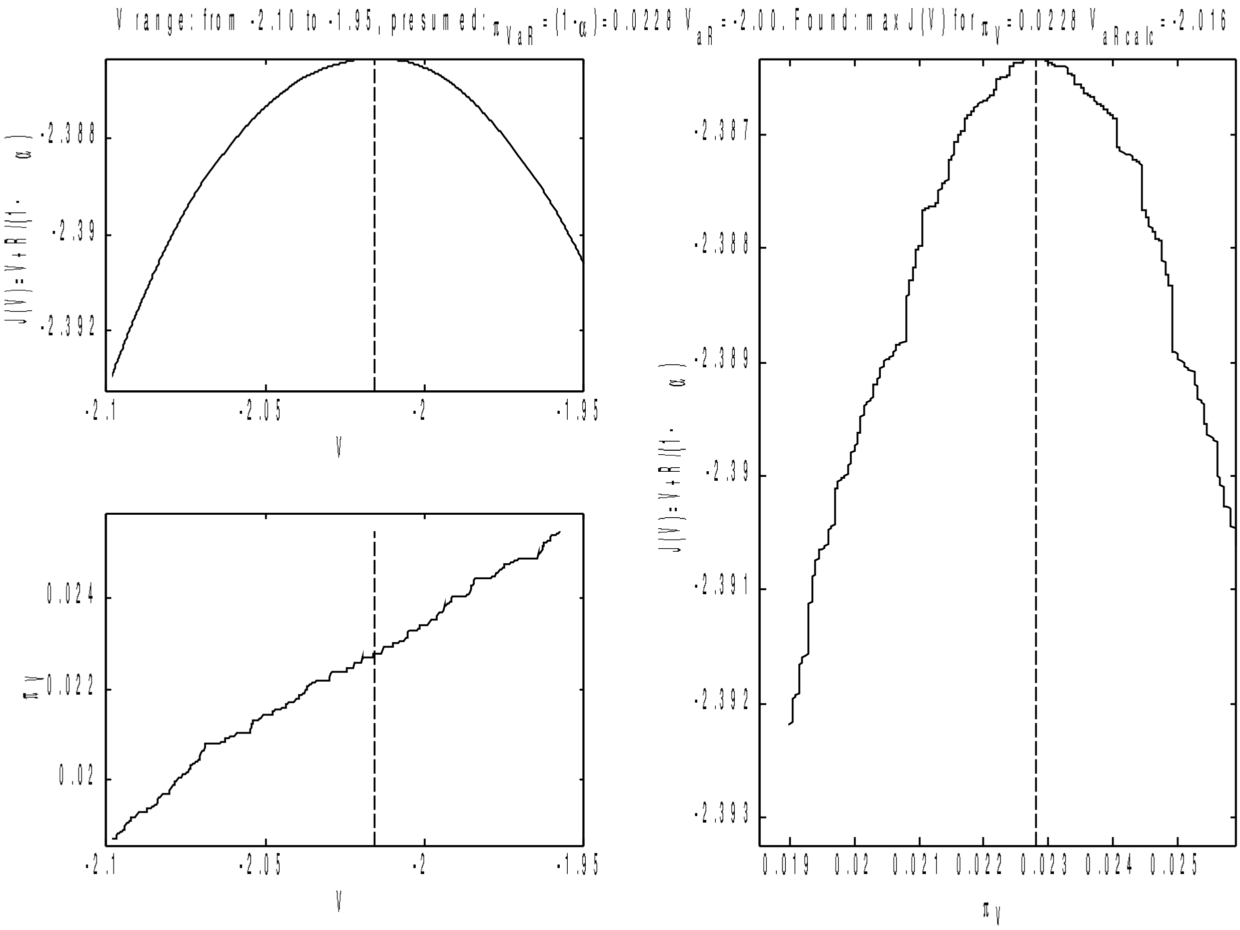

Figure 6.1: Results for V, π_V and function $J(V)$ with the use of random sampling techniques (Monte Carlo calculations)

The results presented in figures 6.1 and 6.2 were calculated numerically for 20000 samples (Monte Carlo calculations) of Gaussian distribution $N(0,1)$, where results were computed for $dV = 0.001$.

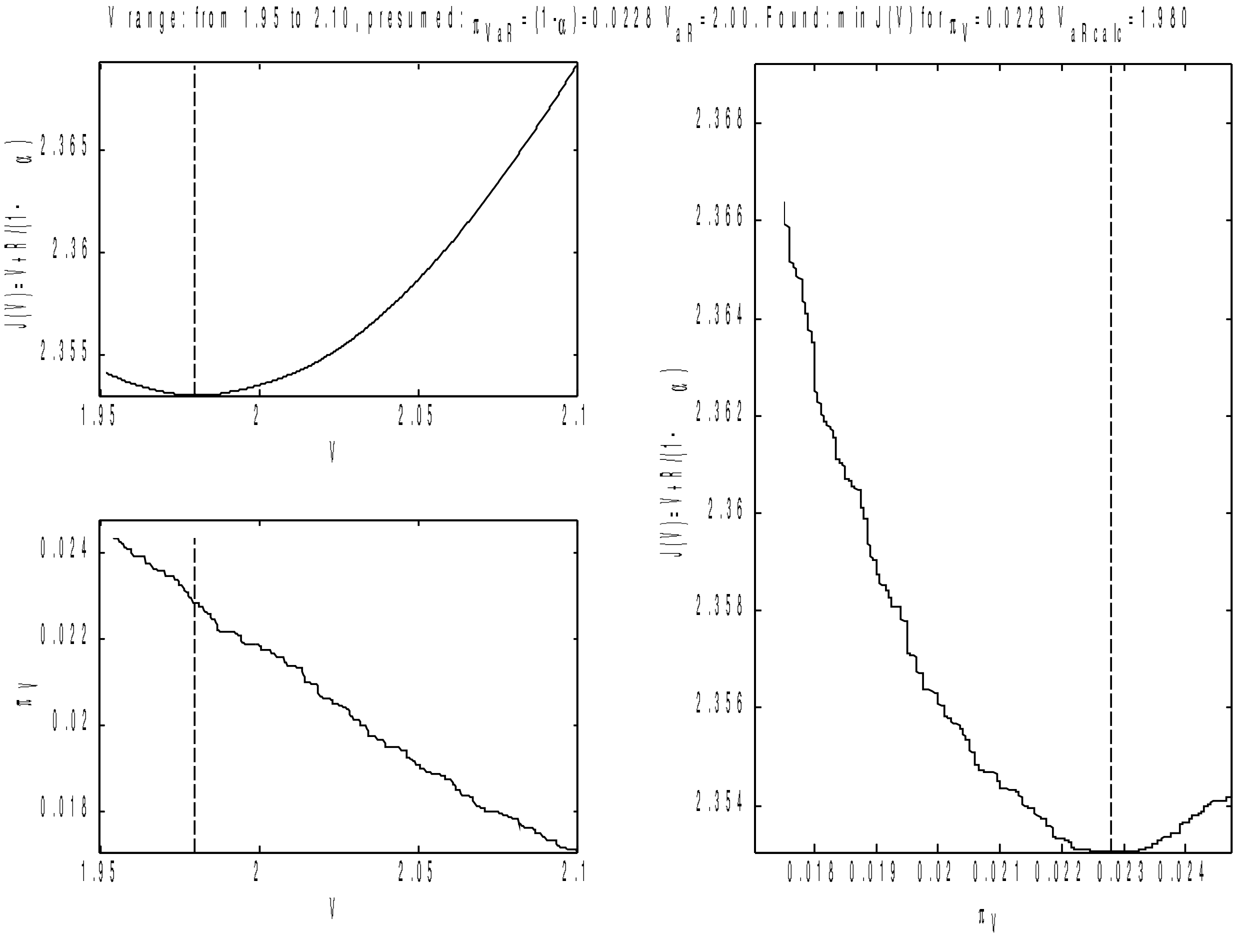

Figure 6.2: Results for V, π_V and function $J(V)$ with the use of random sampling techniques (Monte Carlo calculations)

In the figure 6.1 relation between V, π_V and function $J(V)$ has been shown for V (VaR) range from -2.10 to -1.95, presumed $\pi_V = 1 - \alpha = 0.0228$ for $VaR = -2.00$. Found: maximum of function $J(V)$ for $\pi_V = 0.0228$ and computed $VaR = -2.016$.

In the figure 6.2 relation between V, π_V and function $J(V)$ has been shown for V (VaR) range from 1.95 to 2.10, presumed $\pi_V = 1-\alpha = 0.0228$ for $VaR = 2.00$. Found: minimum of function $J(V)$ for $\pi_V = 0.0228$ and computed $VaR = 1.980$.

Description of axes for figures 6.1 and 6.2. are as follows:

Chapter 7

Multiple Criteria Optimization Models for Assignment of Supporting Services in Health Care

7.1 Introduction to optimization in healthcare

The assignment of service positions plays an important role in healthcare institutions. Poorly assigned positions in hospital departments or over-employment may result in increased expenses and/or degraded customer service. If too many workers are assigned, capital costs are likely to exceed the desirable value (Brandeau at al., 2004 [24]). The supporting services have a strong impact on performance of healthcare institutions such as hospitals. In hospital departments, the supporting services include financial management, logistics, inventory management, analytic laboratories, etc. This paper presents an application of operations research model for optimal supporting service jobs allocation in a public healthcare institution. The optimality criterion of the problem is to minimize operations costs of a supporting service subject to some specific constraints. The constraints representing specific conditions for resource allocation in a hospital were modified, compared to previous publications (Sawik, 2008b, 2010d, 2010f [106, 121, 123]). The overall problem is formulated as a mixed integer program in the literature known as the assignment problem (Bertsimas at al., 1997; Burrkard at al., 2008; Nemhauser at al., 1999 [18, 28, 89]). The binary decision variables represent the assignment of people to various services. This paper shows practical usefulness of mathematical programming approach to optimization of supporting services in healthcare institutions. The results of some computational experiments modeled after a real data from a selected Polish hospital are reported.

7.2 Data used for computations

The real data from a selected Polish public healthcare institution from a one month period were used for computations. The data include 17 supporting service hospital departments, in which there are 74 types of supporting service jobs (Sawik, 2008b, 2010d, 2010f [106, 121, 123]). Permanent employment is defined as a percent of permanent post between 25% (0.25) to 100% (1.00) according to the size of a job position (part-time or full time) for a selected job in a selected department. It is possible that a department has four half time permanent employees and this could be for example an equivalent to two full time permanent employments. Supporting service departments in the hospital consist in total of 78.50 permanent employments with 192 workers employed before the optimization. Specific data consists of the average salaries for selected jobs in the departments defined as costs of assignment of workers to jobs. Furthermore, the average amount of money paid monthly for services in each department was used. Additional parameters include the number of permanent employments in each department and the size of permanent employments (i.e. 0.25, 0.50, 0.75, 1.00) for each job defined as partial or full time. In addition, the minimum number of permanent employments for each job in each department was given, and the maximal number of positions which can be assigned to a single worker.

Table 7.1 presents the number of workers and service jobs in the hospital departments and the total number of workers in all departments before the optimization.

Table 7.2 shows the number of types of permanent employments and the maximum amount of money paid for services in the hospital departments before optimization.

7.3 Problem Formulation

Mathematical programming approach deals with optimization problems of maximizing or minimizing a function of many variables subject to inequality and equality constraints and integrality restrictions on some or all of the variables. In particular, 0-1 variables represent binary choice. Therefore, the model presented in this paper is defined as a mixed integer programming problem. Suppose there are m people and p jobs, where $m \neq p$. Each job must be done by at least one person; also, each person can do at least, one job. The cost of person i doing job k is $\bar{c_{ik}}$. The problem objective

Table 7.1: Number of workers and service jobs in the hospital departments before optimization

Supporting service departments	Number of workers	Number of jobs
Central Heating Department	16	5
Power Department	15	3
Medical Bottled Gases Department	6	2
Ventilation & Air-condition Department	8	4
Heating & Air-condition Department	11	4
Distribution Department	6	3
Medical Equipment Department	8	4
Technical Department	11	5
Economy Department	21	5
Hospital Pharmacy	20	11
Sterilization Department	27	5
Material Monitoring Department	13	5
Information Department	7	4
Business Executive Department	8	5
Technical Executive Department	4	4
Law Regulation Department	7	3
Attorneys-at-law Department	4	2
Number of workers in all department	192	74

Table 7.2: Number of Permanent employments and the maximum amount of money paid for services in the hospital departments before optimization

Supporting service departments	Number of types of permanent employments	Amount of money paid for services [PLN]
Central Heating Department	5	29250
Power Department	3	31050
Medical Bottled Gases Department	2	11400
Ventilation & Air-condition Department	4	16650
Heating & Air-condition Department	4	21200
Distribution Department	3	13600
Medical Equipment Department	4	17500
Technical Department	5	20950
Economy Department	5	31360
Hospital Pharmacy	11	43400
Sterilization Department	5	41500
Material Monitoring Department	5	27150
Information Department	4	16100
Business Executive Department	5	15450
Technical Executive Department	4	7150
Law Regulation Department	7	16100
Attorneys-at-law Department	2.5	7950
Money paid for services in all departments	78.5	367760

is to assign the people to the jobs so as to minimize the total cost of completing all of the jobs.

The optimality criterion of the defined problem is to minimize operations costs of a supporting service subject to some specific constraints. The constraints represent specific conditions for resource allocation in a hospital. The overall problem is formulated as a modified assignment problem. The decision variables represent the assignment of people among various services. Compared to previously published papers (Sawik 2008b, 2010d, 2010f [106, 121, 123]).

Table 7.3: Notations for mathematical models M28, M29, M30

		Indices
i	=	worker, $i \in I = \{1, \ldots, m\}$
j	=	supporting service hospital department, $j \in J = \{1, \ldots, n\}$
k	=	type of supporting service job, $k \in K = \{1, \ldots, q\}$
		Input parameters
$\bar{c_{ik}}$	=	cost of assignment of a worker i to job k (i.e. monthly salary)
$\bar{C_j}$	=	maximal monthly budget for salaries in a department j
$\bar{e_k}$	=	size of permanent (partial or full time) employment for job k (i.e. $\bar{e_k}$= 0.25 or 0.50 or 0.75 or 1.00)
$\bar{E_j}$	=	maximal number of permanent employments in a department j
$\bar{h_{jk}}$	=	minimal number of permanent employments for job k in a department j
$\bar{\beta_i}$	=	weight of the objective functions $\bar{f_i}, i = 1, 2, 3$
γ	=	small positive value
$\bar{f_1^{opt}}$	=	ideal solution value of number of workers selected for an assignment to any job in any department
$\bar{f_2^{opt}}$	=	ideal solution value of operational costs of the supporting services
$\bar{f_3^{opt}}$	=	ideal solution value of number of permanent employments for all jobs in all departments

7.4 Optimization Models

The problem of optimal assignment is formulated as a single objective (**M28**, **M29**) or triple objective mixed integer program (**M30**), which allows commercially available

7.4.1 Computational Results

In this section numerical examples and some computational results are presented to illustrate possible applications of the proposed formulations of integer programming of optimal assignment of service positions. Selected problem instances with the examples are modeled on a real data from a Polish hospital. In the computational experiments the historical data is considered. Computational time takes only a fraction of a second to find optimal solution if any exists. The computational results for models **M28** and **M29** have been obtained using AMPL with solver CPLEX 9.1 on computer Compaq Presario 1830 with Pentium III; RAM 512MB. The computational experiments for model **M30** have been performed using AMPL programming language (Fourer, 1990 [47]) and the CPLEX v.11 solver (with the default settings) on a laptop with IntelCore 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM.

Table 7.5 shows the comparison of CPU time requirement for finding optimal solution (models **M28** and **M29**) with the use of constraint (7.7) or (7.12).

Table 7.5: Comparison of computational results (models **M28** and **M29**) with alternative constraints

Scenario	Operational costs [PLN]	Number of assigned workers	MIP simplex iteration	CPU	Constraint
A	153,251	77	2	10.49	(7.7)
A	153,251	77	0	12.25	(7.12)
B	209,751	108	3	12.46	(7.7)
B	209,751	108	0	12.08	(7.12)
C	248,951	131	3	9.17	(7.7)
C	248,951	131	0	12.80	(7.12)
D	311,651	166	4	7.47	(7.7)
D	311,651	166	0	7.47	(7.12)

*CPU seconds for proving optimality on Pentium III, RAM 512MB / CPLEX v.9.1

Table 7.6 presents the reference point values of parameters for computational experiments with the method optimization model (**M30**) and the size of adjusted problem.

Table 7.7 presents comparison of computational results (model **M30**) with alternative scenarios.

Table 7.6: The values of parameters for computational experiments and the size of adjusted problem

Scenario	$f_1^{\bar{opt}}$	$f_2^{\bar{opt}}$	$f_3^{\bar{opt}}$	All variables	Binary variables	Constraints
A	70	150000	75	4448	3188	566
B	110	200000	105	4416	3156	526
C	130	250000	120	4416	3156	526
D	160	300000	155	4404	3144	512

$\gamma = 0.01$	$\beta_1 = 0.33 \cdot 1000$	$\beta_2 = 0.34$	$\beta_3 = 0.33 \cdot 1000$

Table 7.7: Comparison of computational results (model **M30**) with alternative scenarios

Scenario	δ	Number of workers	Operational costs [PLN]	Number of permanent employments	MIP simplex iterations	B-&-B nodes	CPU seconds
A	1320.00	74	147,201	71.50	297	6	0.265
B	3842.51	109	211,302	105.75	161	0	0.202
C	330.00	131	248,952	121.75	433	0	0.296
D	1802.51	162	305,302	159.00	310	0	0.171

*CPU seconds for proving optimality on IntelCore 2 Duo T9300 processor running at 2.5GHz, 4GB RAM / CPLEX v.11

In tables 7.5 and 7.7 column ”MIP simplex iteration” shows the number of mixed integer programming simplex iterations until the solution is presented. Column ”B-&-B nodes” shows the number of searched nodes in the branch and bound tree until presented solution.

In table 7.8 the number of workers assigned to the supporting service hospital departments and the number of permanent employments is presented (model **M30**).

As it has been recommended by the hospital managers four different scenarios of the assignment have been implemented. In scenario A, a minimal number of people is employed in each supporting service department so that each type of a job has at least one worker assigned. This rule is implemented in input parameter $\bar{h_{jk}}$. In scenario B at least two workers were assigned to each job. Scenario C secured the level of supporting service workers. In each department there are at least two workers assigned to each job, but for some special cases, more than two workers are assigned to each job. Finally, scenario D presents the optimal assignment of workers to jobs with a high service level

Table 7.8: Number of workers assigned and number of permanent employments in departments

Assignment of workers in departments according to scenario								
Supporting service departments	workers	permanent employments	workers	permanent employments	workers	permanent employments	workers	permanent employments
Number of workers and permanent employments in all Departments	74 (A)	71.5 (A)	109 (B)	105.75 (B)	131 (C)	121.75 (C)	162 (D)	159 (D)
Attorneys-at-law Department	2 (A)	1.5 (A)	3 (B)	2.5 (B)	3 (C)	2 (C)	3 (D)	2.5 (D)
Law Regulation Department	3 (A)	3 (A)	5 (B)	5 (B)	5 (C)	4 (C)	5 (D)	5 (D)
Technical Executive Department	4 (A)	3.5 (A)	4 (B)	3.5 (B)	4 (C)	3.5 (C)	4 (D)	3.5 (D)
Business Executive Department	5 (A)	5 (A)	6 (B)	6 (B)	6 (C)	6 (C)	7 (D)	7 (D)
Information Department	4 (A)	3.5 (A)	5 (B)	4.5 (B)	5 (C)	4.5 (C)	6 (D)	5.5 (D)
Material Monitoring Department	5 (A)	5 (A)	7 (B)	7 (B)	8 (C)	8 (C)	11 (D)	11 (D)
Sterilization Department	5 (A)	5 (A)	8 (B)	8 (B)	14 (C)	13.5 (C)	21 (D)	21 (D)
Hospital Pharmacy	11 (A)	10.5 (A)	15 (B)	14.5 (B)	17 (C)	14.5 (C)	19 (D)	18.5 (D)
Economy Department	5 (A)	5 (A)	9 (B)	9 (B)	14 (C)	13.5 (C)	18 (D)	18 (D)
Technical Department	5 (A)	5 (A)	8 (B)	8 (B)	8 (C)	7.5 (C)	9 (D)	9 (D)
Medical Equipment Department	4 (A)	4 (A)	6 (B)	5.75 (B)	6 (C)	5.75 (C)	7 (D)	6.5 (D)
Distribution Department	3 (A)	3 (A)	5 (B)	5 (B)	5 (C)	4.5 (C)	5 (D)	5 (D)
Heating and Air-condition Department	4 (A)	4 (A)	5 (B)	5 (B)	7 (C)	7 (C)	9 (D)	9 (D)
Ventilation and Air-condition Department	4 (A)	3 (A)	6 (B)	6 (B)	6 (C)	5.5 (C)	7 (D)	7 (D)
Medical Bottled Gases Department	2 (A)	2 (A)	3 (B)	3 (B)	4 (C)	3.5 (C)	5 (D)	5 (D)
Power Department	3 (A)	3 (A)	5 (B)	5 (B)	8 (C)	8 (C)	12 (D)	12 (D)
Central Heating Department	5 (A)	4.5 (A)	9 (B)	8 (B)	11 (C)	10.5 (C)	14 (D)	13.5 (D)

*Scenarios A, B, C & D considered subject to hospital authority requirements with all currently employed workers. The results obtained have indicated the problem of over-employment in the hospital.

7.4.2 Conclusions for Multiple Criteria Optimization Models for Assignment of Supporting Services in Health Care

Operations research techniques, tools and theories have long been applied to a wide range of issues and problems in healthcare. This paper proves the practical usefulness of mathematical programming approach to optimization of supporting service in a hospital. The results of computational experiments modeled after a real data from a hospital in Lesser Poland indicate that the number of hired workers can be reduced in almost all departments of the hospital.

The proposed modified multi-objective assignment problem and a reference point approach can be easily implemented for management of supporting services in another institution, not only healthcare. Obtained results consist of the monthly expenses for salaries, the number of workers and the amount of permanent employments needed for jobs in all considered supporting service departments.

Computational time takes only a fraction of a second to find the optimal solution because of a relatively small size of the input data. Presented optimization model is NP-hard, but computable. Implementation of reference point method ensures to obtain results with non-dominated set of solutions. The global optimums for considered three objective functions are presented.

Chapter 8

Computational Experiments

All presented in this chapter computational experiments were conducted on a laptop with IntelCore 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM. For the implementation of portfolio models, the AMPL programming language and the CPLEX v.11 solver (with the default settings) were applied.

In this section the strength of $CVaR$ and VaR approach and MIP models is demonstrated on computational examples. The data sets for the example problems were based on historic daily portfolios of the Warsaw Stock Exchange.

In the computational experiments the five levels of the confidence level was applied $\alpha \in \{0.99, 0.95, 0.90, 0.75, 0.50\}$, and for the weighted-sum program the subset of non-dominated solutions were computed by parametrization on λ.

Table 8.1 presents the influence of different parameters on CPU run time.

Table 8.1: Problem parameters vs. Central Processing Unit run time

$1 - \alpha$ increases	CPU decreases
VaR increases	CPU increases
m increases	CPU increases

Figure 8.1 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.99$.

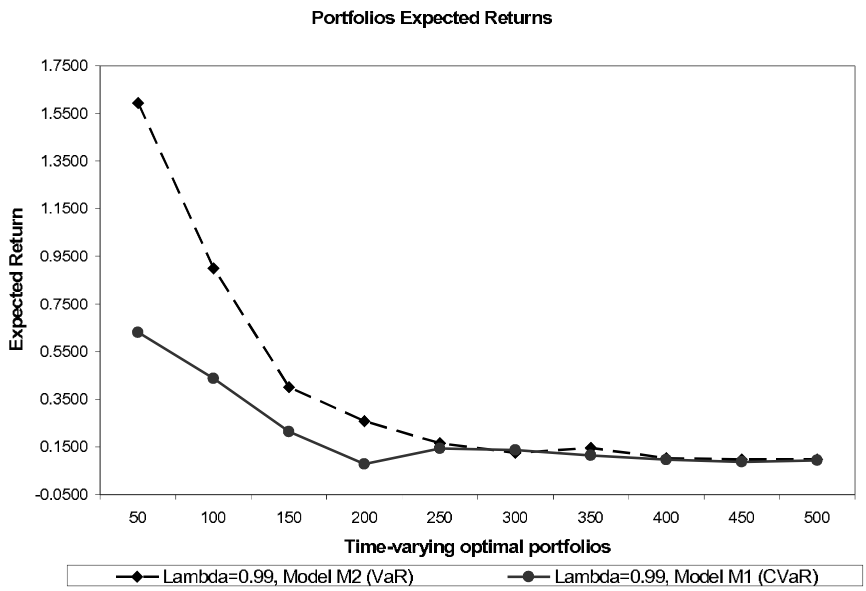

Figure 8.1: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.99$

Figure 8.2 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.95$.

Figure 8.3 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.90$.

Figure 8.4 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.75$.

Figure 8.5 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.50$.

Figure 8.6 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.25$.

Figure 8.7 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.10$.

Figure 8.8 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.05$.

Figure 8.9 shows comparison of computed expected returns for time-varying optimal portfolios for models M1 and M2 with objective weight $\lambda = 0.01$.

Figure 8.10 presents computed Value-at-Risk (VaR) in model M1, which in model M2 is an input parameter $VaR = -1$ (for presented computational example) - $\lambda = 0.99$.

Figure 8.11 presents computed Value-at-Risk (VaR) in model M1, which in model

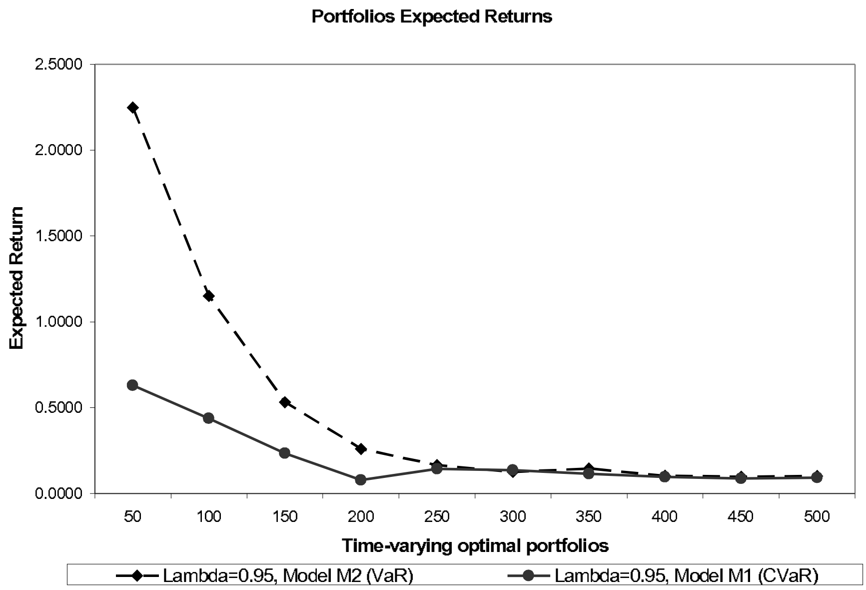

Figure 8.2: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.95$

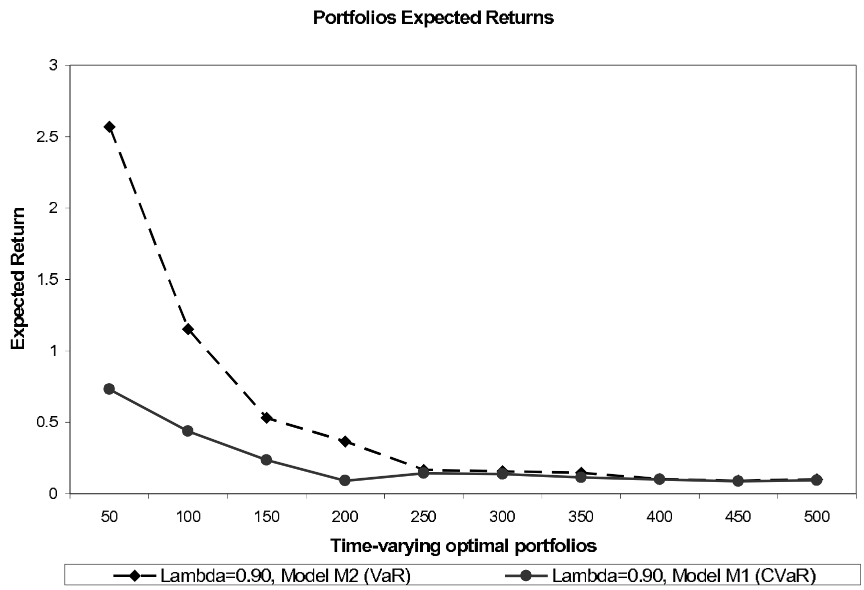

Figure 8.3: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.90$

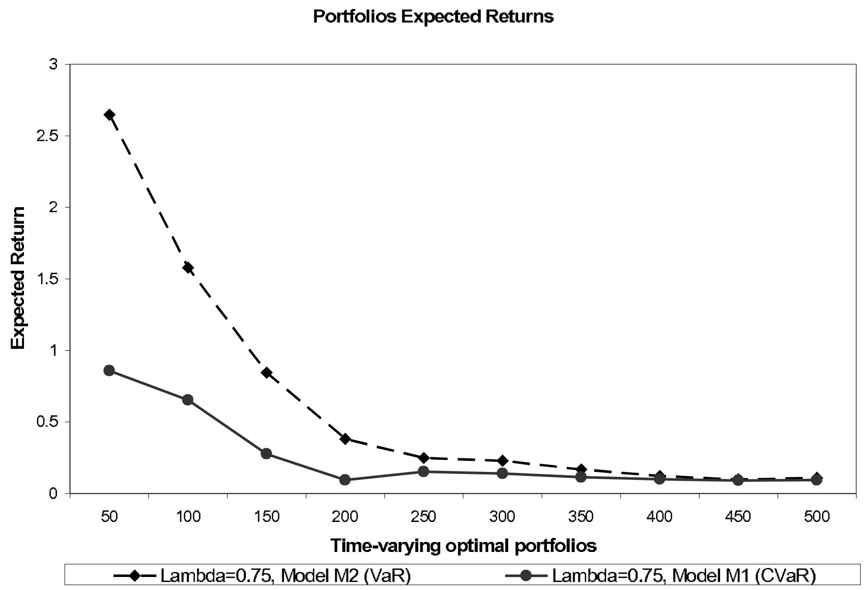

Figure 8.4: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.75$

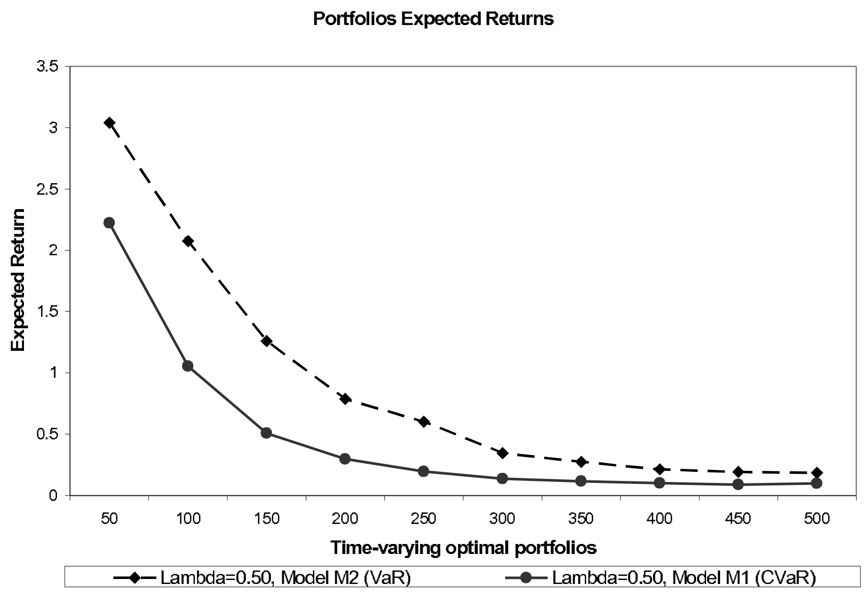

Figure 8.5: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.50$

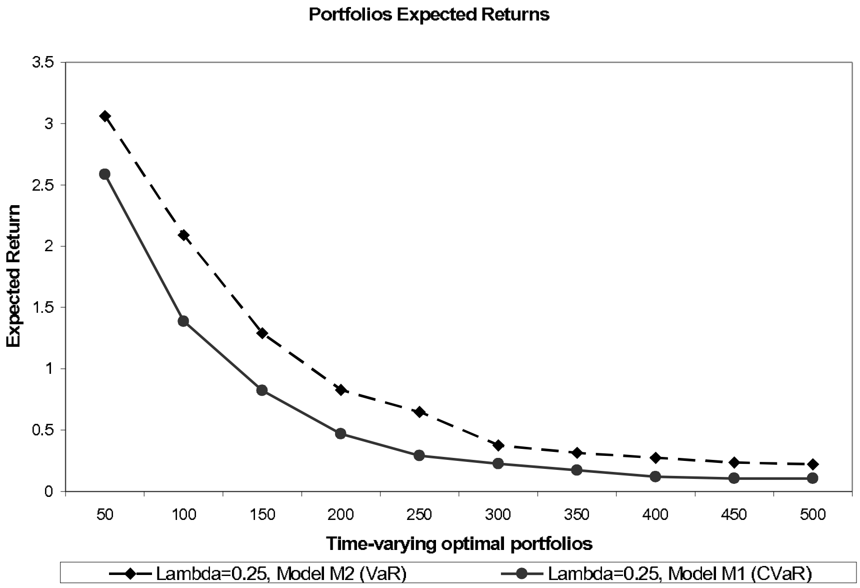

Figure 8.6: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.25$

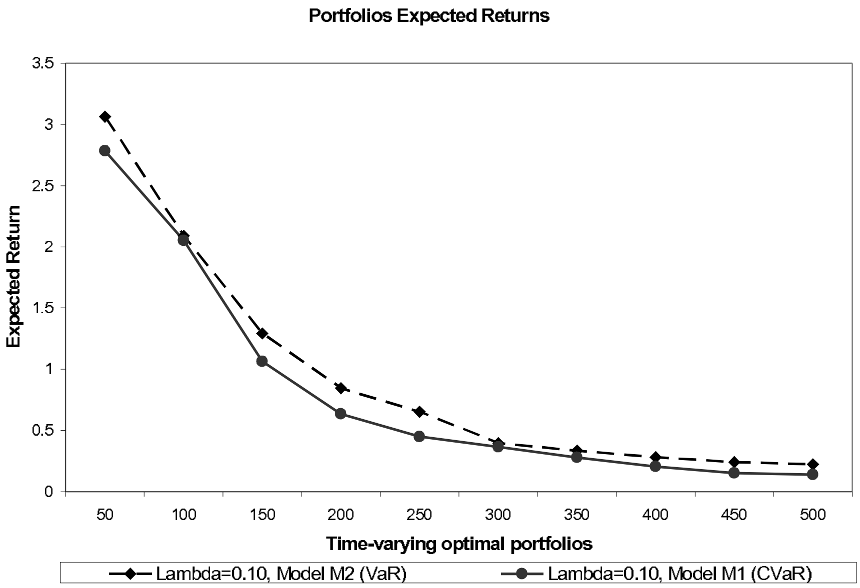

Figure 8.7: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.10$

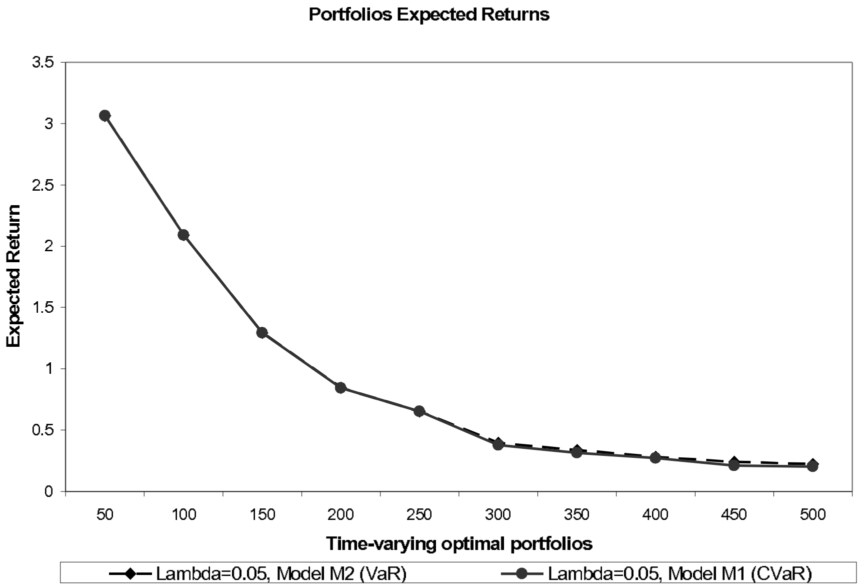

Figure 8.8: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.05$

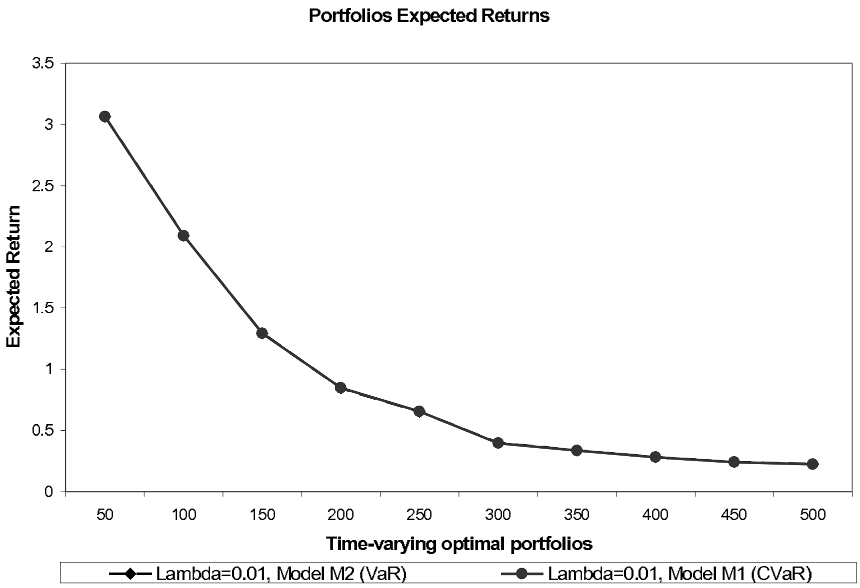

Figure 8.9: Computed expected returns for time-varying optimal portfolios with $\lambda = 0.01$

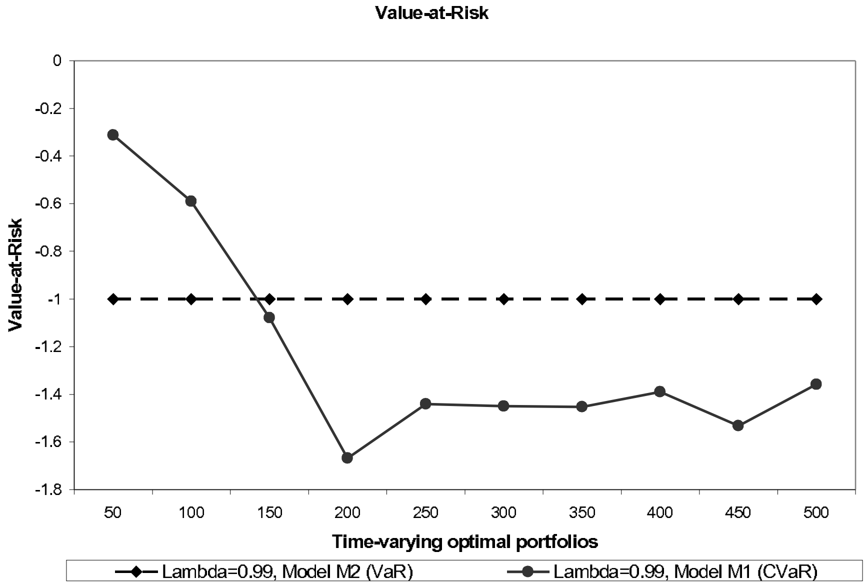

Figure 8.10: Value-at-Risk for time-varying optimal portfolios with $\lambda = 0.99$

M2 is an input parameter $VaR = -1$ (for presented computational example) - $\lambda = 0.50$.

Figure 8.12 presents computed Value-at-Risk (VaR) in model M1, which in model M2 is an input parameter $VaR = -1$ (for presented computational example) - $\lambda = 0.05$.

Figure 8.13 shows comparison of α (upper curve) and π_V (lower curve) - $\lambda = 0.99$, where $\pi_V = 1 - \alpha$ (see Chapter 6).

Figure 8.14 shows comparison of α and π_V - $\lambda = 0.50$.

Figure 8.15 shows comparison of α and π_V - $\lambda = 0.05$.

Figure 8.16 presents comparison of number of assets in optimal portfolio - $\lambda = 0.99$.

Figure 8.17 presents comparison of number of assets in optimal portfolio - $\lambda = 0.50$.

Figure 8.18 presents comparison of number of assets in optimal portfolio - $\lambda = 0.05$.

Figure 8.19 shows comparison of computational times - $\lambda = 0.99$.

Figure 8.20 shows comparison of computational times - $\lambda = 0.50$.

Figure 8.21 shows comparison of computational times - $\lambda = 0.05$.

Figure 8.22 presents the Pareto Frontier for time-varying model **M1** - $CVaR$ vs. expected portfolio return for $\alpha = 0.99$.

Figure 8.23 presents the Pareto Frontier for time-varying model **M2** - α vs. expected portfolio return for $VaR = -1$.

Figure 8.24 presents the Pareto Frontier for time-varying model **M3** with cutting constraint - matrix of covariance for selected portfolio vs. expected portfolio return.

Figure 8.25 presents the Pareto Frontier for time-varying model **M3** without cutting

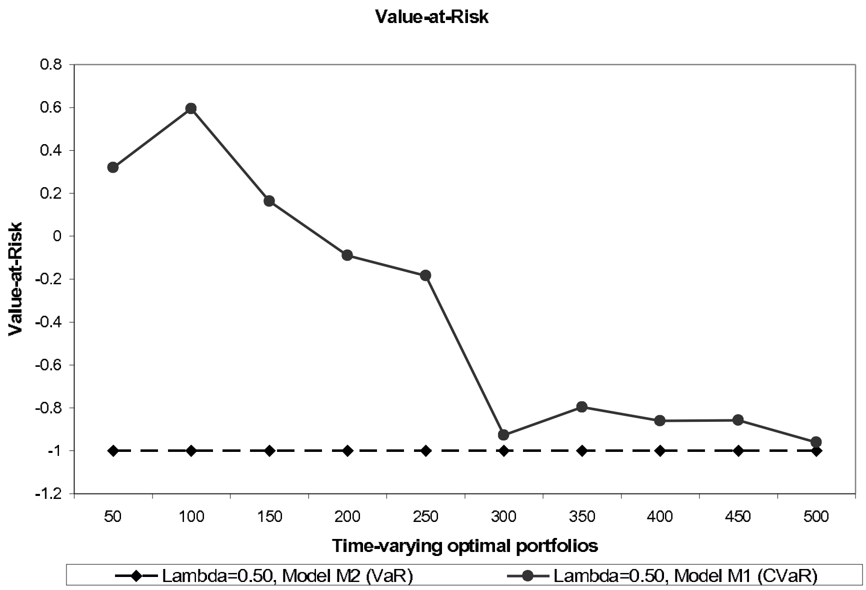

Figure 8.11: Value-at-Risk for time-varying optimal portfolios with $\lambda = 0.50$

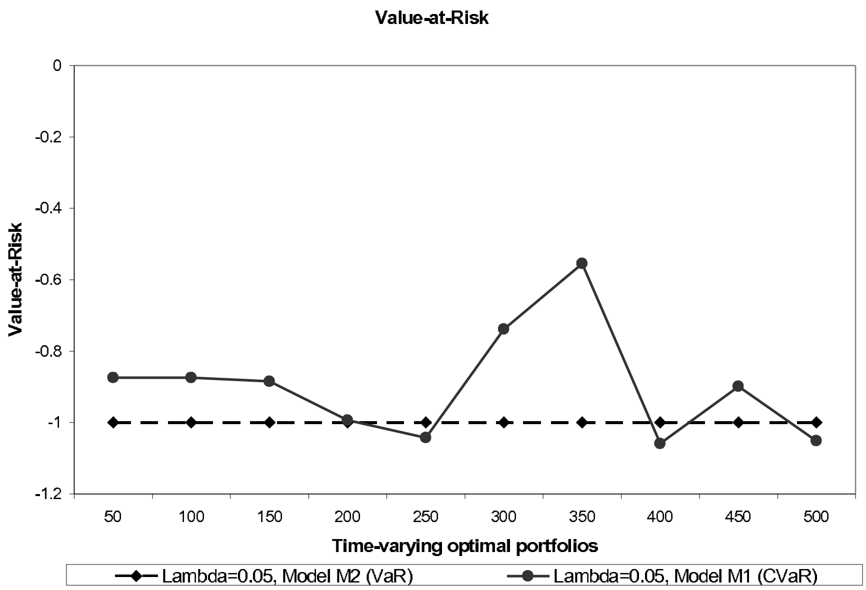

Figure 8.12: Value-at-Risk for time-varying optimal portfolios with $\lambda = 0.05$

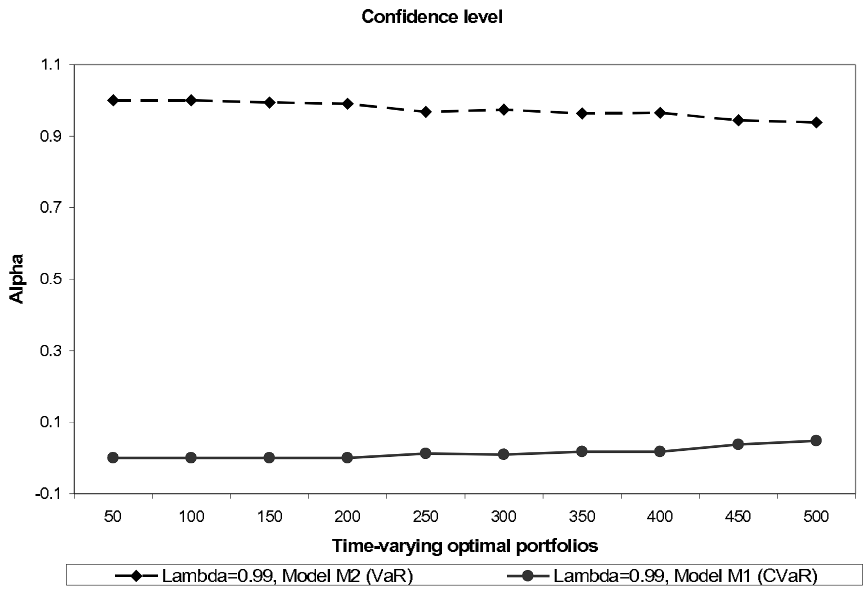

Figure 8.13: α and π_V for time-varying optimal portfolios with $\lambda = 0.99$

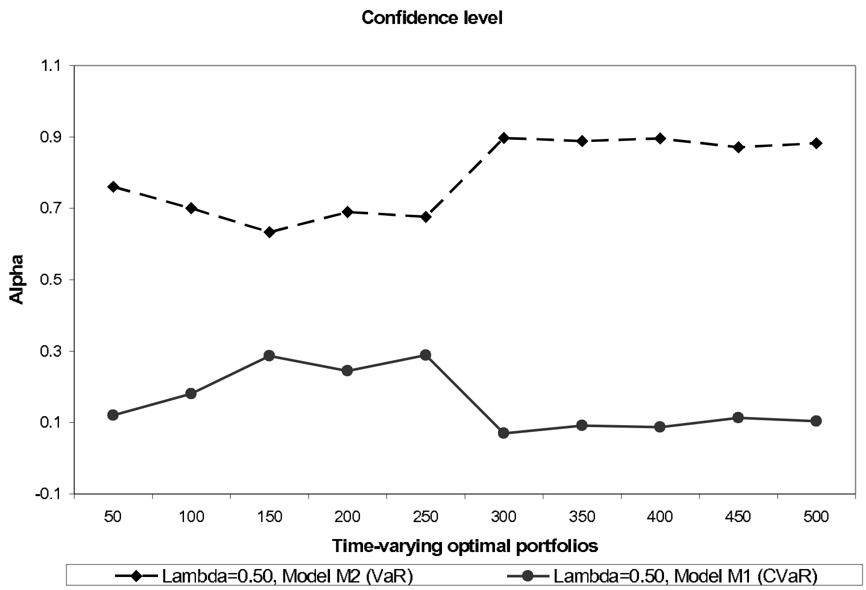

Figure 8.14: α and π_V for time-varying optimal portfolios with $\lambda = 0.50$

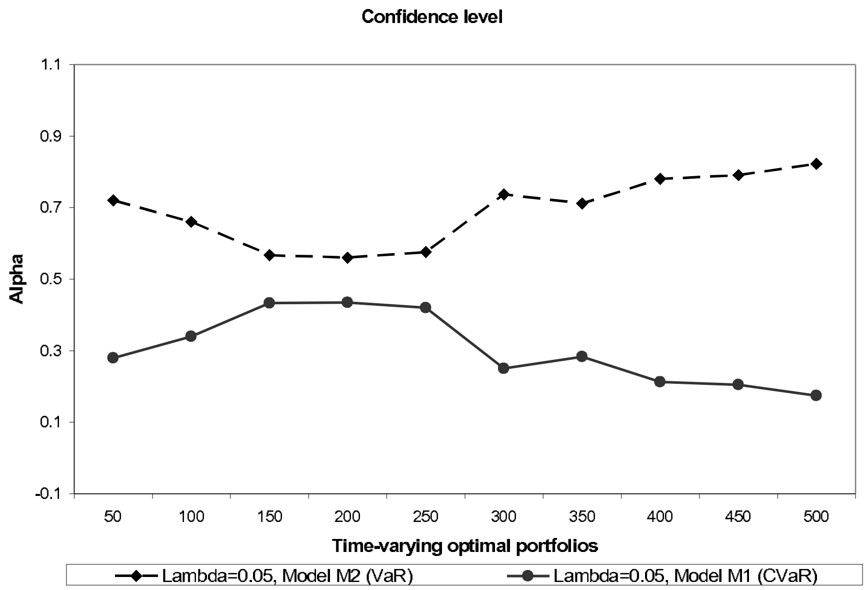

Figure 8.15: α and π_V for time-varying optimal portfolios with $\lambda = 0.05$

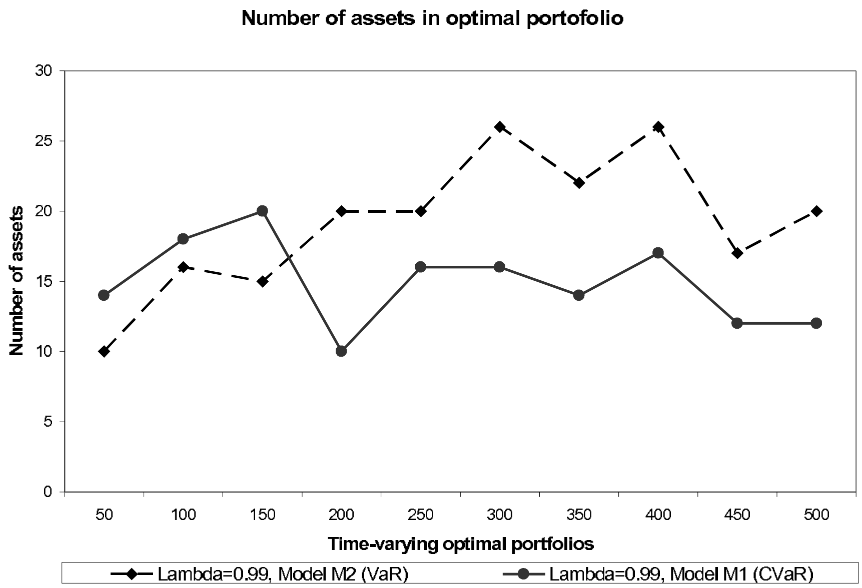

Figure 8.16: Number of selected assets for time-varying optimal portfolios with $\lambda = 0.99$

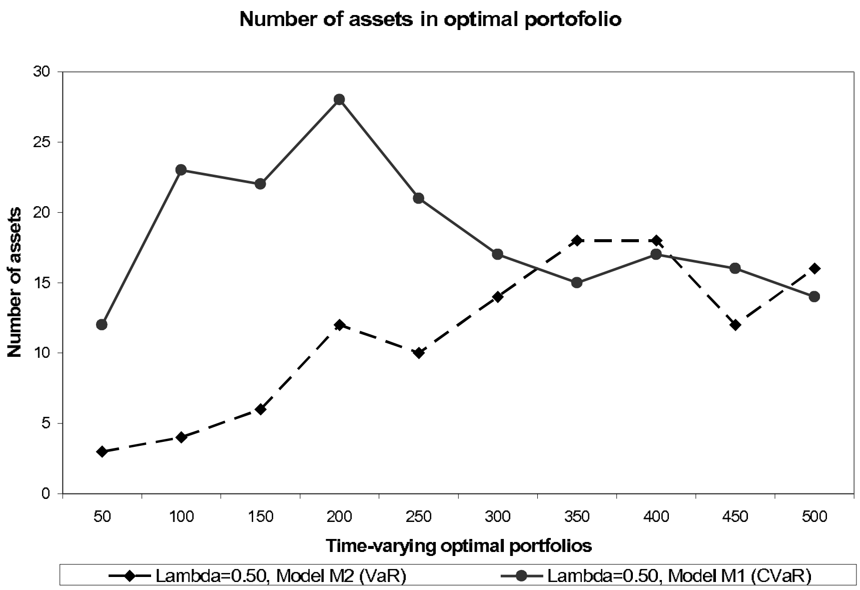

Figure 8.17: Number of selected assets for time-varying optimal portfolios with $\lambda = 0.50$

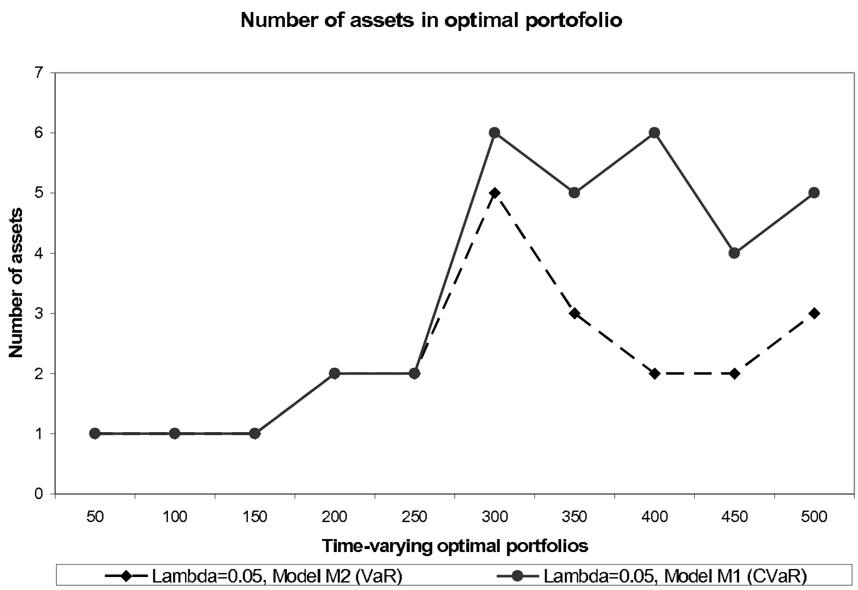

Figure 8.18: Number of selected assets for time-varying optimal portfolios with $\lambda = 0.05$

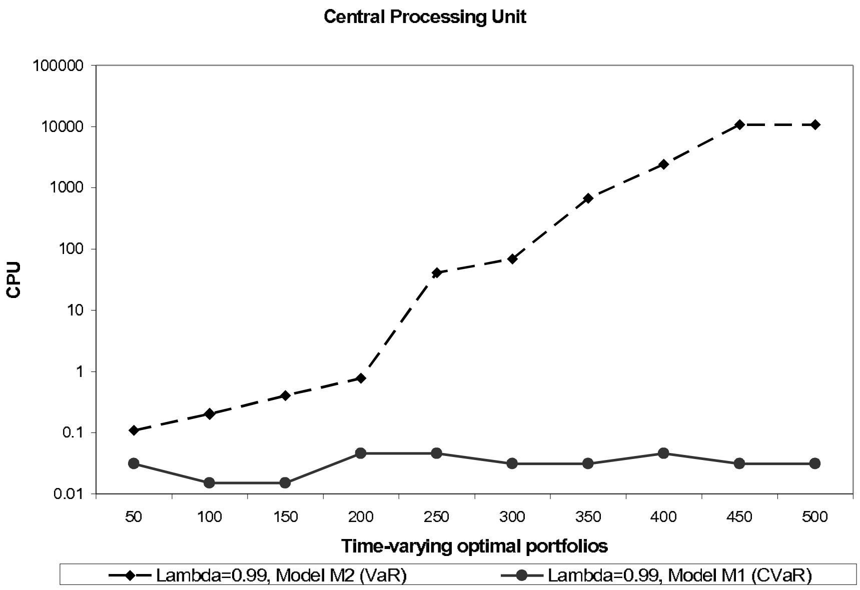

Figure 8.19: Computational times for time-varying optimal portfolios with $\lambda = 0.99$

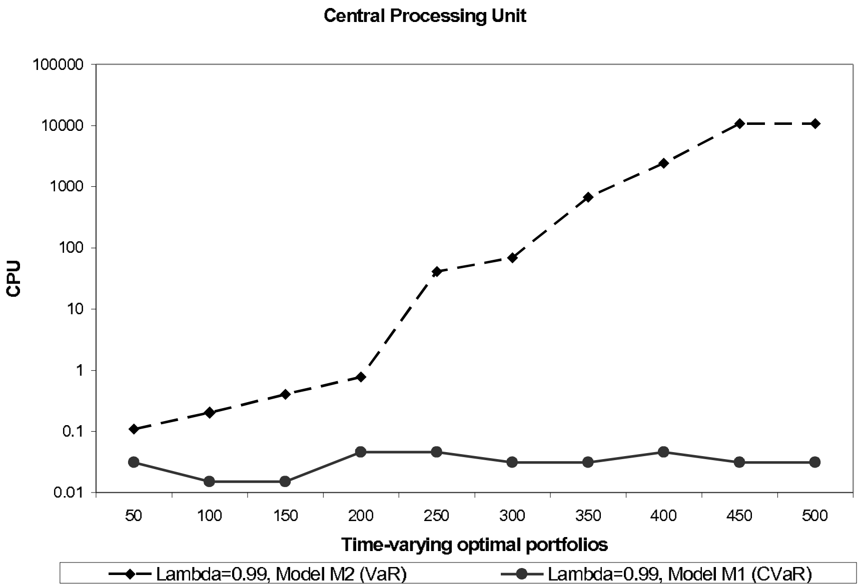

Figure 8.20: Computational times for time-varying optimal portfolios with $\lambda = 0.50$

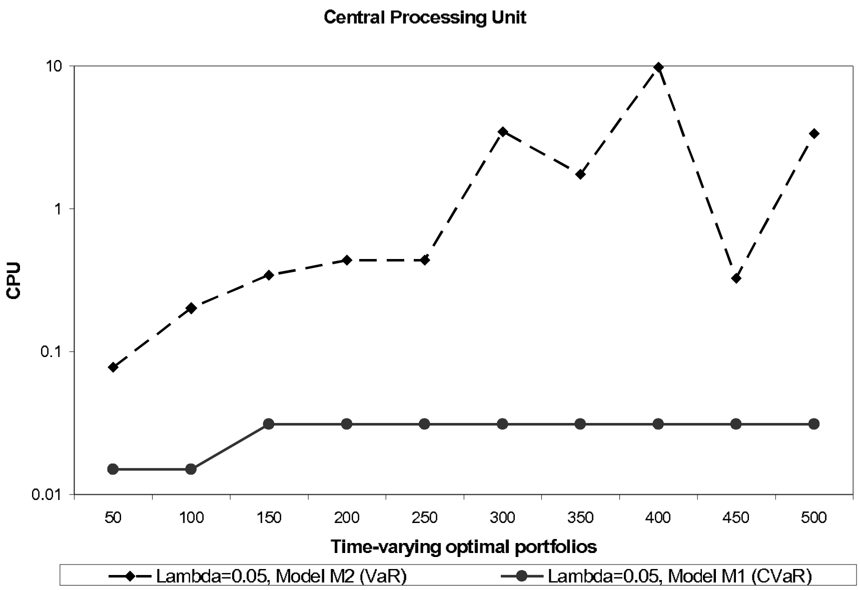

Figure 8.21: Computational times for time-varying optimal portfolios with $\lambda = 0.05$

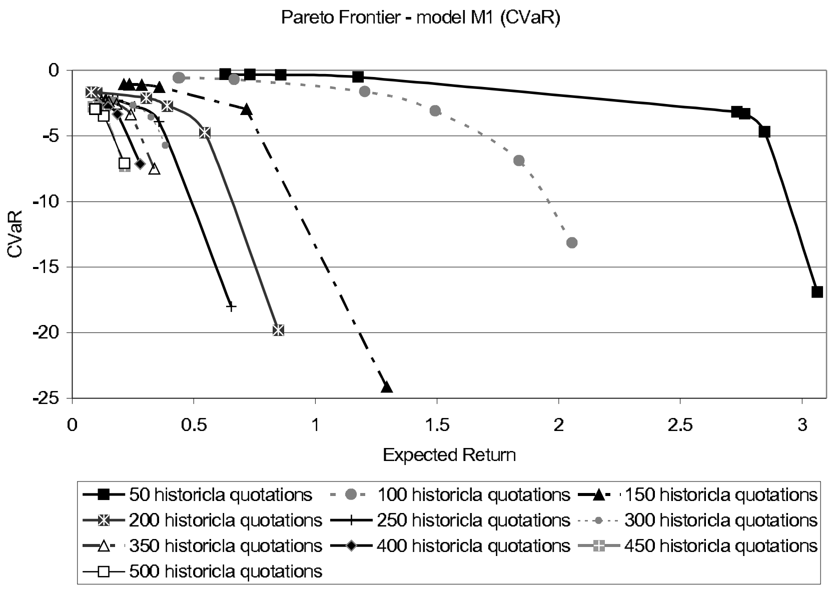

Figure 8.22: Pareto Frontier for time-varying model **M1**

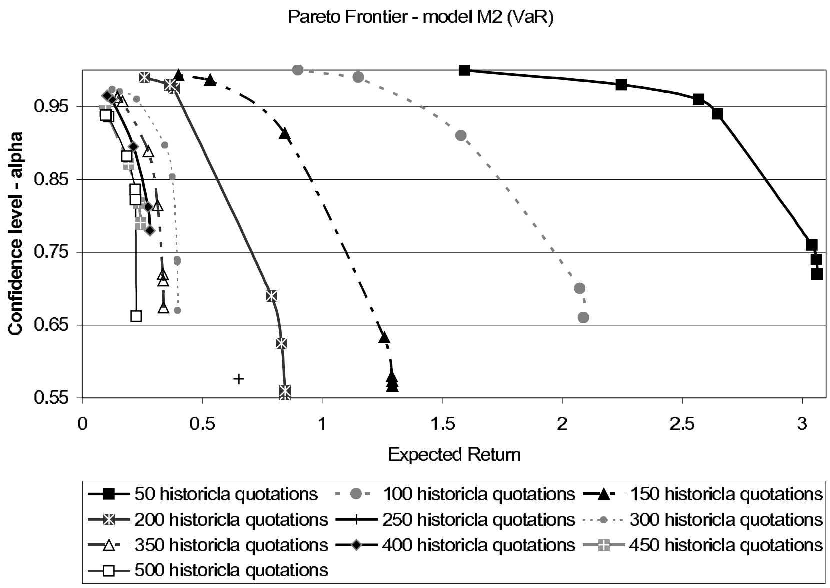

Figure 8.23: Pareto Frontier for time-varying model **M2**

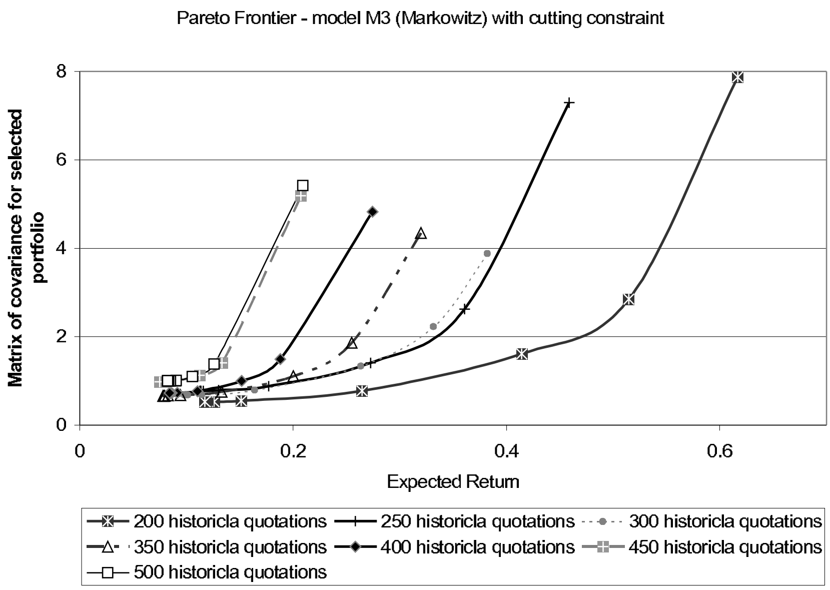

Figure 8.24: Pareto Frontier for time-varying model **M3** with cutting constraint

constraint - matrix of covariance for selected portfolio vs. expected portfolio return.

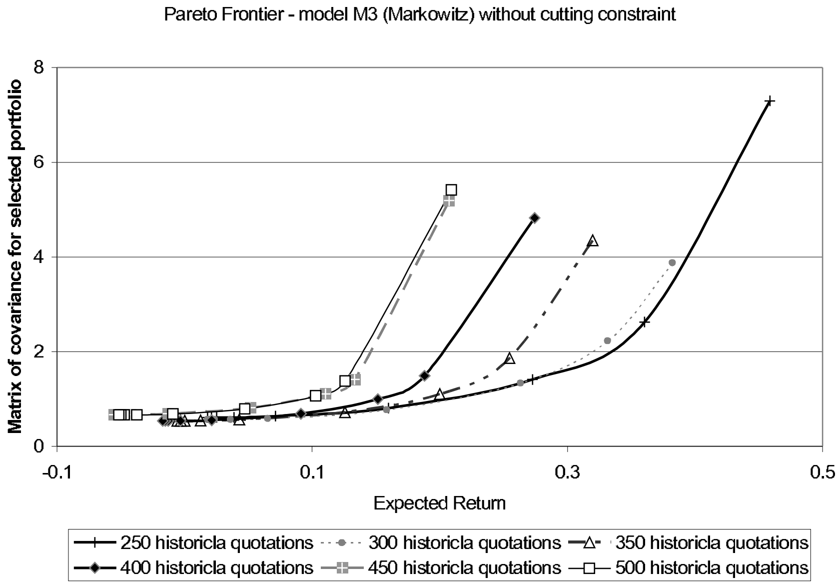

Figure 8.25: Pareto Frontier for time-varying model **M3** without cutting constraint

Figure 8.26 presents comparison of portfolio expected return (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

Figure 8.27 presents comparison of computational times of solved portfolios (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

Figure 8.28 presents comparison of number of stocks (securities) in portfolios (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

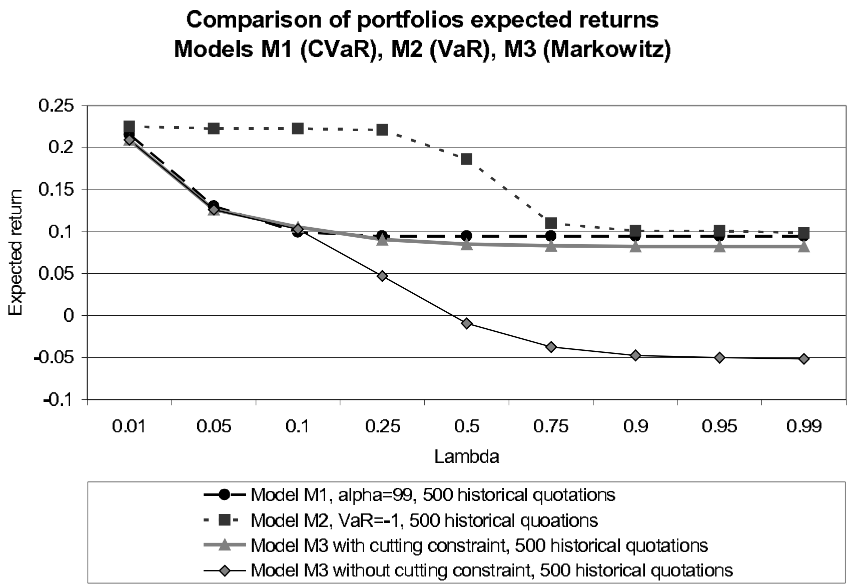

Figure 8.26: Comparison of portfolio expected return (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

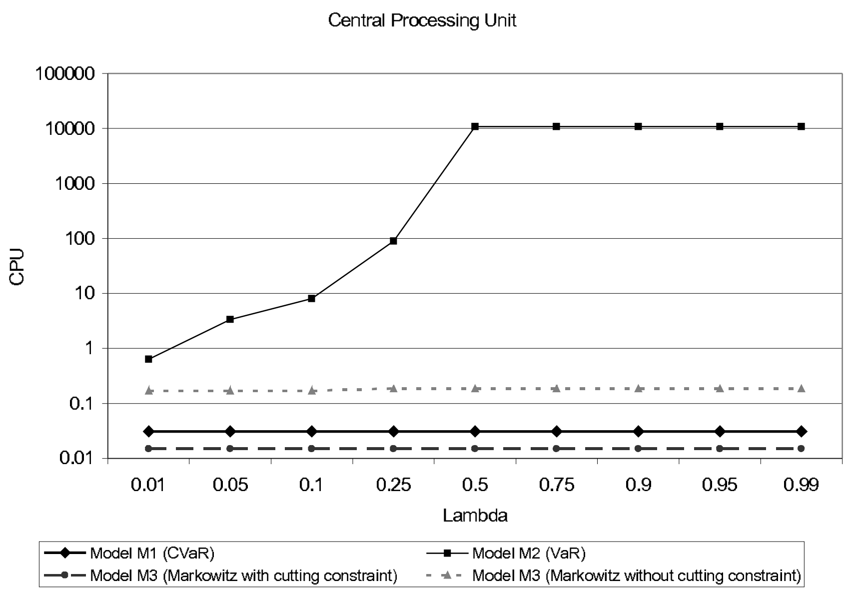

Figure 8.27: Comparison of computational times of solved portfolios (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

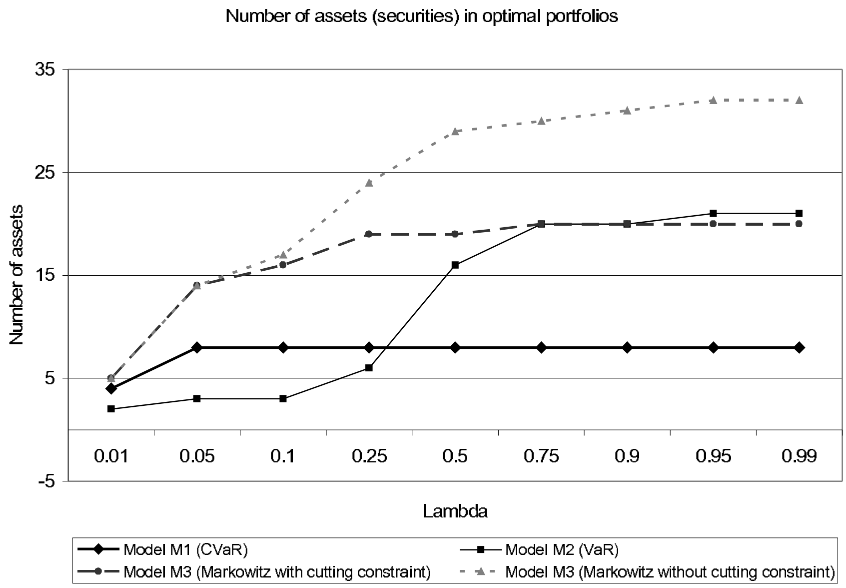

Figure 8.28: Comparison of number of stocks (securities) in portfolios (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

Chapter 9

Summary and Conclusion

9.1 Conclusions

The purpose of this dissertation was to present and compare the weighting, lexicographic and reference point approach and the corresponding mixed integer programming formulations for the multi-criteria portfolio optimization problem.

In particular, the research efforts were concentrated on mixed integer programming formulations. The need for solving multi-objective portfolio optimization models by mixed integer programming has been illustrated for the portfolio models with Value-at-Risk (VaR) as a risk measure, as well as, when the number of assets (investment ventures) is one of the optimality criteria. An alternative, multi-objective portfolio optimization problems was formulated with Conditional Value-at-Risk ($CVaR$) as a risk measure or with symmetric measure of risk - covariance (variance) of return - as in Markowitz portfolio.

The proposed multi-objective portfolio models were constructed with the expected return as a performance measure and the expected worst-case return as a risk measure, using Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$). These measures have allowed for the evaluation of worst-case return and shaping of the resulting return distribution through the selection of the optimal portfolio. The mathematical programming models were constructed and solved using weighting, lexicographic and reference point approach. The presented portfolio models have been single-, bi- and triple-objectives and the optimization criteria considered are risk, return and number of stocks.

The main research problem considered in this Ph.D. dissertation was how to find the best multi-objective portfolio formulation with risk. The additional research problem was to find the relation between the optimization results with Value-at-Risk solved by mixed integer programming and the results of optimization obtained with the use of linear and quadratic programming portfolio models (Conditional Value-at-Risk, Markowitz).

Computational experiments have been conducted for multi-criteria portfolio models of stock exchange investments. The number of selected securities for input data varies from 46 to 240 assets. The historical stocks quotations came from the period from March 10th, 1997 to February 2nd, 2009. This time period includes data from the increase of stock exchange quotations, as well as the economic crisis period. The considered number of data in historical time series is from 500 to 3000 days with assets quoted each day in the whole historical horizon. The portfolios were optimized in an increased time window, which was helpful in evaluating the results of optimization (time-varying optimal portfolio).

The multi-criteria portfolio optimization models with Conditional Value-at-Risk ($CVaR$) as a risk measure can be used to support on-line stock market investments, since the computational times required to find the optimal solution is relatively short, regardless of the size of the input data. The presented models provide a decision maker with a tool for evaluating the relationship between expected and worst-case returns.

The results obtained from computational experiments proved, that multi-objective portfolio optimization models with Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) could be used to shape the distribution of portfolio returns in a favorable way for a decision maker. The portfolios obtained with both methods (mixed-integer or linear programming) were often similar, which have shown their capability of solving the corresponding problems. It means that a suboptimal portfolio for the integer program with Value-at-Risk (VaR) as optimality criterion can be found by solving the corresponding linear program for the portfolio problem with Conditional Value-at-Risk ($CVaR$) as an optimality criterion. The proposed scenario-based portfolio optimization problems under uncertainty, formulated as a single- or multi-objective mixed integer program were solved using commercially available software (AMPL/CPLEX) for mixed integer programming.

The nature of the portfolio problem focuses on a compromise between the construction of objectives, constraints and decision variables in a portfolio and the problem complexity of the implemented mathematical models. There is always a trade off between computational time and the size of an input data, as well as the type of mathematical programming formulation (linear or mixed integer).

The computational results obtained by modeling the decision criteria (e.g. lexicographically choosing one objective function with the highest priority) in constructed multi-objective portfolio optimization models, could be used by a decision maker for evaluation of his/her investment strategy. It is easy to compare obtained optimal (ideal) solution values of the selected objectives with a real investment situation in the stock market.

In addition to the multi-objective approach for portfolio optimization of securities (e.g. stocks) from stock exchanges presented in this dissertation, the selected multi-objective mixed integer programming models are shown for supporting services in medical care institutions, based on assignment problem.

The proposed portfolio optimization models formulated by mixed integer programming can be effectively implemented in decision support systems for the bi- and triple-objective portfolio optimization, in which variance of return from the risky ventures (investments) was replaced with VaR or $CVaR$.

The scenario-based portfolio optimization problems under uncertainty, formulated as a single- or multi-objective mixed integer program have been easily solved using commercially available software, like AMPL Programming Language with use of CPLEX solver for mixed integer programming.

The computational experiments modeled on a real data from the Warsaw Stock Exchange have indicated that the approach is capable of finding proven optimal solutions for all real-world problems considered, in a reasonable computation time using commercially available software for mixed integer programming.

The total computation time ranges from a few seconds to minutes or even hours depending on the number of historical quotations in the optimization problem and type of optimization problem formulation.

The portfolio optimization models with $CVaR$ could be used for supporting on-line stock market investments, since computational times required for finding optimal solutions are relatively short, regardless of the size of input data for computations (e.g. more than 200 stocks with 3000 quotations).

The computational experiments show that the proposed solution approach based on mixed integer programming models provides the decision maker with a simple tool for evaluating the relationship between the expected and the worst-case portfolio return.

The decision maker can assess the value of portfolio return and the risk level, and can decide how to invest in a real life situation comparing with the ideal (optimal) portfolio solutions. A risk-aversive decision maker wants to maximize the $CVaR$. Since the amount by which losses in each scenario exceed VaR has been constrained of being positive, the presented models try to increase VaR and hence positively impact the objective functions. However, large increases in VaR may result in more historic portfolios (scenarios) with tail return, counterbalancing this effect. The concave efficient frontiers illustrate the trade-off between the $CVaR$ and the expected return of the portfolio.

In all cases the CPU time increases when the confidence level decreases. The number of securities selected for the optimal portfolio for all the models varies between 1 and more than 50 assets. Those numbers show very little dependence on the confidence level α and the size of historical portfolio used as an input data.

The most important results presented in the PhD dissertation are:

- Based on the recent, available publications on portfolio optimization methods, the critical analysis of the literature was made.
- The computational experiments are based on the real data from the time of the increase of stock exchange quotations as well as from the economic crisis period.
- A review of multi-criteria optimization methods for portfolio optimization with the comparison of obtained results was performed.
- The portfolios were optimized in the increased time window, which enabled the evaluation of the results of optimization (time-varying optimal portfolio).
- The comparison of obtained results for portfolio optimization models with symmetric (covariance of return) and percentile measures (VaR and $CVaR$) of risk was presented.
- The single- and multi-period portfolio problems formulations were proposed.
- The formulation of multi-objective portfolio problems was presented with the consideration of different optimization criteria.
- In addition, selected multi-objective mixed integer programming models were shown for supporting services in medical care institutions, based on assignment problem, which could be also transformed into a portfolio problem.

9.2 Future research

In the future research it is planned to perform the multi-objective portfolio optimization for a multi-market portfolio with a different level of risk at each market of ventures. Furthermore it is intended to improve mixed integer programming approach for a multi-period, multi-criteria portfolio optimization by applying some methods from graphs and game theory. The main idea is to define vector of historical optimal portfolio at time t as a vertex in a graph, in which edges will be defined as possible changes of stocks in new optimal portfolio at time $t+1$. This method could lead to an improved combinatorial algorithm for finding optimal portfolio in a multi-period environment. The multi period portfolio optimization over a rolling planning horizon can be enhanced with the addition of short-selling variables [1].

Some application of the game theory (see e.g. von Neumann and Morgenstern, 1944 [90]) will be helpful to find non-dominated optimal solution for multi-objective portfolio optimization, when stocks in optimal portfolio are considered as players who compete for amount of money to be invested in selected stock.

Finally, for a comparison it is intended to conduct some computational experiments using the models developed and the input data from the other markets such as NYSE, ZURICH SMI and, if possible, or from the industry or services, e.g. medical supporting services. The supporting services have a strong impact on performance of health-care institutions such as hospitals. For example in a hospital, the supporting services include financial management, logistics, inventory management, analytic laboratories, etc. Switching from stock exchange to supporting services the optimal portfolio of service positions in selected department of a hospital - could be considered with a dual objective to reach both high quality and low cost of service.

[1] Going short on security (i.e. stock) j means that the corresponding share value x_j is negative: $x_j < 0$. In this way, the signs of the portfolio shares $x_1, ..., x_n$ are not restricted anymore (Wanka and Gohler, 2001 [150])

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Appendix - More Computational Examples

In this section numerical examples and some computational results are presented to illustrate possible applications of the proposed formulations of linear and mixed integer programming of portfolio problems. Selected problem instances with the examples are modeled on a real data from the Warsaw Stock Exchange.

In the computational experiments the historical data is considered. Computational time range is from a few seconds to minutes or even hours. The computational experiments have been performed using:

- AMPL with solver CPLEX v.9.1 on a PC Pentium III, RAM 512MB
- AMPL with solver CPLEX v.9.1 on a PC Pentium IV, RAM 512MB
- AMPL with solver CPLEX v.11 on a PC IntelCore 2 Duo T9300 with 2,5GHz, RAM 4GB

Figure 9.1 presents the solution results for maximization of expected portfolio (model **M22**) return for 3500 historical quotations, divided into 14 multi-period intervals, for the multi-period portfolio (model **M22**) with $1-\alpha = 0.05$ and in figure 9.2 the solution results are indicated for $1-\alpha = 0.10$.

For comparison table 9.1 shows the results for single-period portfolio optimization.

Figure 9.3 presents the solution results (model **M22**) for maximization of expected portfolio return for 4020 historical quotations, divided into 20 multi-period intervals, for the multi-period portfolio with $1-\alpha = 0.1$ and in figure 9.4 shows the number of securities in computed portfolios for each historical multi-period interval k with $1-\alpha = 0.1$. Figures 9.5–9.6 shows results of problem **M22** with $1-\alpha = 0.5$. For comparison figure

Security	1	2	3	4	5
ALMAMARKET	0.000000	0.000000	0.000000	0.000000	**0.465391**
BANKBPH	0.000000	0.000000	0.000000	0.000000	**0.018391**
BOS	0.000000	0.000000	0.000000	0.000000	**0.023972**
BRE	0.000000	**0.177618**	0.000000	0.000000	0.000000
BZWBK	0.000000	0.000000	0.000000	**0.160697**	0.000000
COMPLAND	0.000000	0.000000	0.000000	**0.272527**	0.000000
DEBICA	0.000000	0.000000	**0.639504**	0.000000	0.000000
ECHO	0.000000	0.000000	0.000000	**0.342196**	**0.025669**
ELEKTROEX	0.000000	0.000000	0.000000	**0.026355**	0.000000
FORTISPL	0.000000	0.000000	**0.070774**	0.000000	0.000000
IRENA	0.000000	**0.106870**	0.000000	0.000000	0.000000
JUTRZENKA	0.000000	0.000000	0.000000	**0.082132**	0.000000
KABLE	**0.003649**	0.000000	0.000000	0.000000	0.000000
KROSNO	**0.000693**	0.000000	**0.034640**	0.000000	0.000000
MILLENNIUM	0.000000	**0.024055**	0.000000	0.000000	0.000000
MOSTALEXP	0.000000	**0.101409**	0.000000	0.000000	0.000000
MOSTALZAB	0.000000	0.000000	0.000000	0.000000	**0.005751**
PROCHEM	0.000000	0.000000	0.000000	0.000000	**0.030507**
PROCHNIK	**0.303749**	0.000000	0.000000	0.000000	**0.125751**
PROVIMROL	0.000000	0.000000	0.000000	**0.116094**	0.000000
RAFAKO	0.000000	0.000000	**0.255083**	0.000000	0.000000
RELPOL	0.000000	0.000000	0.000000	0.000000	**0.106429**
SWARZEDZ	0.000000	**0.001966**	0.000000	0.000000	0.000000
ZYWIEC	0.000000	0.000000	0.000000	0.000000	**0.198139**

Security	6	7	8	9	10
05VICT	**0.084712**	0.000000	0.000000	0.000000	0.000000
13FORTUNA	0.000000	0.000000	0.000000	**0.003436**	0.000000
7BULLS	0.000000	**0.268007**	0.000000	0.000000	0.000000
APATOR	0.000000	0.000000	0.000000	**0.003248**	**0.030493**
BIOTON	0.000000	0.000000	0.000000	0.000000	0.000000
BORYSZEW	0.000000	0.000000	0.000000	0.000000	**0.091807**
CERSANIT	0.000000	0.000000	0.000000	**0.023096**	0.000000
COMARCH	0.000000	**0.731993**	0.000000	0.000000	0.000000
EFEKT	**0.195242**	0.000000	0.000000	0.000000	0.000000
ELEKTRIM	**0.026971**	0.000000	0.000000	0.000000	0.000000
ENAP	0.000000	0.000000	0.000000	0.000000	**0.442006**
ENERGOPN	0.000000	0.000000	0.000000	**0.186252**	0.000000
FARMACOL	0.000000	0.000000	0.000000	**0.021771**	0.000000
FORTISPL	**0.165586**	0.000000	0.000000	0.000000	0.000000
HYDROBUD	0.000000	0.000000	0.000000	0.000000	**0.185723**
IGROUP	0.000000	0.000000	**0.140239**	0.000000	0.000000
INDYKPOL	**0.510657**	0.000000	0.000000	0.000000	0.000000
IRENA	0.000000	0.000000	**0.403446**	0.000000	0.000000
KOMPAP	0.000000	0.000000	0.000000	**0.091613**	0.000000
KOPEX	0.000000	0.000000	0.000000	**0.667304**	0.000000
LPP	0.000000	0.000000	0.000000	0.000000	**0.115837**
NOVITA	0.000000	0.000000	**0.050955**	0.000000	0.000000
ORBIS	0.016833	0.000000	0.000000	0.000000	0.000000
PEKAO	0.000000	0.000000	0.000000	**0.001956**	0.000000
PROCHNIK	0.000000	0.000000	0.000000	0.000000	**0.070045**
SANWIL	0.000000	0.000000	0.000000	**0.001324**	0.000000
SUWARY	0.000000	0.000000	0.000000	0.000000	**0.064090**
TIM	0.000000	0.000000	**0.287144**	0.000000	0.000000
TUEUROPA	0.000000	0.000000	**0.118216**	0.000000	0.000000

Security	11	12	13	14
04PRO	0.000000	0.000000	0.000000	**0.006966**
08OCTAVA	0.000000	0.000000	0.000000	**0.017490**
ALCHEMIA	0.000000	**0.135950**	**0.519819**	0.000000
BEEFSAN	0.000000	**0.110792**	0.000000	0.000000
BORYSZEW	**0.286390**	0.000000	0.000000	0.000000
BUDOPOL	0.000000	0.000000	**0.059895**	0.000000
CAPITAL	0.000000	0.000000	**0.050812**	0.000000
ELMONTWAR	0.000000	0.000000	**0.072730**	0.000000
ELZAB	0.000000	0.000000	0.000000	**0.009828**
ENAP	0.000000	**0.014319**	0.000000	0.000000
ENERGOPLD	0.000000	0.000000	0.000000	**0.144623**
ENERGOPOL	0.000000	0.000000	0.000000	**0.536332**
GANT	0.000000	0.000000	**0.103931**	0.000000
IBSYSTEM	**0.115000**	0.000000	0.000000	0.000000
LZPS	**0.598610**	0.000000	0.000000	0.000000
MASTERS	0.000000	0.000000	0.000000	**0.056304**
MIDAS	0.000000	0.000000	0.000000	**0.085664**
POLNORD	0.000000	0.000000	0.000000	**0.048126**
PONARFEH	0.000000	0.000000	0.000000	**0.012163**
PPWK	0.000000	**0.058896**	0.000000	0.000000
PROCHNIK	0.000000	0.000000	**0.014402**	0.000000
SANWIL	0.000000	**0.165758**	0.000000	0.000000
SKOTAN	0.000000	0.000000	**0.178412**	0.000000
STALPROD	0.000000	0.000000	0.000000	**0.078952**
SUWARY	0.000000	0.000000	0.000000	**0.003552**
TIM	0.000000	**0.124986**	0.000000	0.000000
TUEUROPA	0.000000	**0.091813**	0.000000	0.000000
TUP	0.000000	**0.297487**	0.000000	0.000000

Porfolio Return for t=1	0.211097	aVaR
Porfolio Return for t=2	0.146049	0.05
Porfolio Return for t=3	0.535848	
Porfolio Return for t=4	0.683758	rMin
Porfolio Return for t=5	0.341943	-100
Porfolio Return for t=6	0.219909	
Porfolio Return for t=7	0.706013	rVar
Porfolio Return for t=8	0.454442	-3
Porfolio Return for t=9	0.357582	
Porfolio Return for t=10	0.528617	m
Porfolio Return for t=11	1.241630	250
Porfolio Return for t=12	0.789012	
Porfolio Return for t=13	1.686130	p
Porfolio Return for t=14	1.280990	0.004

Portfolio Return	**9.183020**
MIP simplex iteration	**2413180**
B-&-B	**251147**
CPU*	**1231.80**

Figure 9.1: The example of solution results for the multi-period portfolio (model **M22**) with $1 - \alpha = 0.05$

Security	1	2	3	4	5	6	7
BRE	0.000000	**0.078329**	0.000000	0.000000	0.000000	0.000000	0.000000
COMARCH	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	**1.000000**
COMPLAND	0.000000	0.000000	0.000000	**0.151433**	0.000000	0.000000	0.000000
DEBICA	0.000000	0.000000	**1.000000**	0.000000	0.000000	0.000000	0.000000
ECHO	0.000000	0.000000	0.000000	**0.690348**	0.000000	0.000000	0.000000
ENAP	0.000000	0.000000	0.000000	0.000000	0.000000	**0.007303**	0.000000
FORTISPL	0.000000	0.000000	0.000000	0.000000	0.000000	**0.017723**	0.000000
INDYKPOL	0.000000	0.000000	0.000000	**0.128846**	0.000000	**0.945556**	0.000000
IRENA	0.000000	**0.212968**	0.000000	0.000000	0.000000	0.000000	0.000000
KABLE	**0.052426**	0.000000	0.000000	0.000000	**0.018355**	0.000000	0.000000
KROSNO	**0.046504**	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
MILLENNIUM	0.000000	0.000000	0.000000	**0.011496**	0.000000	0.000000	0.000000
MOSTALEXP	0.000000	**0.213030**	0.000000	0.000000	0.000000	0.000000	0.000000
ORBIS	0.000000	0.000000	0.000000	0.000000	0.000000	**0.025835**	0.000000
PROCHNIK	**0.283915**	0.000000	0.000000	0.000000	**0.057279**	0.000000	0.000000
PROVIMROL	0.000000	0.000000	0.000000	**0.017878**	0.000000	0.000000	0.000000
STRZELEC	0.000000	0.000000	0.000000	0.000000	0.000000	**0.003583**	0.000000
ZYWIEC	0.000000	**0.173739**	0.000000	0.000000	**0.132526**	0.000000	0.000000

Security	13	14
04PRO	0.000000	**0.032104**
ALCHEMIA	**0.975494**	0.000000
BUDOPOL	**0.004686**	**0.011644**
CASHFLOW	0.000000	**0.009391**
CENSTALGD	0.000000	**0.021219**
ELMONTWAR	**0.000678**	0.000000
ENERGOPOL	0.000000	**0.748319**
GANT	**0.018533**	**0.058505**
IGROUP	0.000000	**0.005536**
MIDAS	0.000000	**0.052037**
PONARFEH	0.000000	**0.061245**
SKOTAN	**0.000609**	0.000000
	aVaR	
250	**0.1**	**rVar**
quotations		**-3**
	rMin	
	-100	p
		0.004

Security	8	9	10	11	12
ALCHEMIA	0.000000	0.000000	0.000000	0.000000	**0.143398**
APATOR	0.000000	**0.026778**	0.000000	0.000000	0.000000
BEEFSAN	0.000000	0.000000	0.000000	0.000000	**0.068612**
BORYSZEW	0.000000	0.000000	0.000000	**0.382386**	0.000000
ENAP	0.000000	0.000000	**0.676023**	0.000000	**0.006750**
ENERGOPN	0.000000	**0.095058**	0.000000	0.000000	0.000000
HYDROBUD	0.000000	0.000000	**0.006214**	0.000000	0.000000
IBSYSTEM	0.000000	0.000000	0.000000	**0.344998**	0.000000
IGROUP	**0.213368**	0.000000	0.000000	0.000000	0.000000
IRENA	**0.263941**	0.000000	0.000000	0.000000	0.000000
KOPEX	0.000000	**0.839259**	0.000000	0.000000	0.000000
LPP	0.000000	0.000000	**0.296758**	0.000000	0.000000
LZPS	0.000000	0.000000	0.000000	**0.272616**	0.000000
NOVITA	0.166283	0.000000	0.000000	0.000000	0.000000
SANWIL	0.000000	0.000000	0.000000	0.000000	**0.095295**
SWIECIE	0.000000	**0.038905**	0.000000	0.000000	0.000000
TIM	**0.347096**	0.000000	0.000000	0.000000	**0.073472**
TUEUROPA	**0.009312**	0.000000	0.000000	0.000000	**0.086402**
TUP	0.000000	0.000000	**0.021005**	0.000000	**0.526072**

Porfolio Return for t=1	0.222420
Porfolio Return for t=2	0.243502
Porfolio Return for t=3	0.556498
Porfolio Return for t=4	0.818427
Porfolio Return for t=5	0.415506
Porfolio Return for t=6	0.271328
Porfolio Return for t=7	0.714200
Porfolio Return for t=8	0.460659
Porfolio Return for t=9	0.378635
Porfolio Return for t=10	0.569819
Porfolio Return for t=11	1.240000
Porfolio Return for t=12	0.871209
Porfolio Return for t=13	1.956760
Porfolio Return for t=14	1.389350
Portfolio Return	**10.108300**
MIP simplex iteration	**924273**
B-&-B	**169278**
CPU*	**1155.62**

Figure 9.2: The example of solution results for the multi-period portfolio (model **M22**) with $1-\alpha = 0.10$

Table 9.1: Solution results for the single-period portfolio with $1-\alpha \in \{0.05; 0.10\}$ with $VaR = -3$, $r^{Min} = -100$, $m = 3500$

$1-\alpha$	Portfolio Return	Number of assets	MIP simplex iteration	B-&-B	CPU*
0.05	0.075821	3	6872886	664295	34770
0.10	0.118745	3	3270492	627711	26289

VaR	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical multi-period interval k	Computational results for objective function $\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_j^k$ expected portfolios return													
1	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724
2	0.041199	0.045682	0.025675	0.013863	0.014368	0.007053	0.012762	0.009923	0.009158	0.007047	0.004488	0.002345	0.000657	0.000000
3	0.004459	0.010475	0.010000	0.004197	0.002953	0.005212	0.004380	0.002748	0.002089	0.001896	0.002102	0.001109	0.000352	0.000000
4	0.001279	0.003171	0.004682	0.004754	0.004208	0.002131	0.001760	0.001444	0.002369	0.001641	0.001368	0.000754	0.000152	0.000000
5	0.000207	0.043646	0.006123	0.008315	0.005573	0.005731	0.004260	0.003844	0.001508	0.001819	0.001664	0.000859	0.000103	0.000000
6	0.000000	0.003982	0.005457	0.006786	0.004401	0.005147	0.003306	0.003767	0.002294	0.001364	0.001641	0.000894	0.000205	0.000000
7	0.004345	0.008859	0.004950	0.006502	0.005614	0.005601	0.004815	0.004393	0.002529	0.003418	0.002005	0.001102	0.000080	0.000000
8	0.009675	0.002291	0.001072	0.001407	0.002248	0.003866	0.001949	0.001729	0.002815	0.001885	0.001035	0.000796	0.000081	0.000000
9	0.006826	0.005552	0.005195	0.003615	0.003584	0.002005	0.002783	0.001406	0.000886	0.001003	0.001106	0.000617	0.000070	0.000000
10	0.002178	0.011213	0.016063	0.001892	0.003845	0.006272	0.003331	0.002933	0.004785	0.001851	0.002074	0.000484	0.000185	0.000000
11	0.001073	0.001461	0.002009	0.004250	0.002908	0.004250	0.003126	0.001457	0.001409	0.001821	0.000862	0.000598	0.000148	0.000000
12	0.002227	0.010050	0.003957	0.004558	0.003945	0.004086	0.003528	0.003939	0.002576	0.001476	0.001077	0.000354	0.000228	0.000000
13	0.011015	0.014014	0.009940	0.010538	0.007423	0.006836	0.008030	0.002626	0.003609	0.006006	0.001852	0.001039	0.000207	0.000000
14	0.028984	0.004766	0.005733	0.008194	0.010088	0.004820	0.005008	0.004965	0.002383	0.002486	0.001039	0.002060	0.000155	0.000000
15	0.042169	0.012277	0.012226	0.004751	0.006774	0.009347	0.006124	0.001817	0.003598	0.002434	0.001518	0.000649	0.000330	0.000000
16	0.019164	0.006269	0.002768	0.002042	0.002108	0.007567	0.002387	0.003400	0.004417	0.004489	0.001319	0.000803	0.000229	0.000000
17	0.044785	0.056279	0.033479	0.012656	0.008462	0.002401	0.004627	0.002226	0.005015	0.003556	0.002980	0.001068	0.000185	0.000000
18	0.039349	0.049817	0.013281	0.012195	0.007890	0.007086	0.007165	0.002377	0.008415	0.002656	0.002348	0.001433	0.000304	0.000000
19	0.005587	0.007299	0.011207	0.004093	0.002983	0.004149	0.003649	0.002779	0.002259	0.001667	0.001213	0.000689	0.000157	0.000000
20	0.004941	0.003824	0.003127	0.003007	0.002070	0.002533	0.001972	0.000944	0.000740	0.000493	0.000454	0.000212	0.000045	0.000000

Expected portfolio return for whole investment period $\sum_{k=1}^{t}\left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_j^k\right)$													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.28719	0.31865	0.19467	0.13534	0.11917	0.11382	0.10269	0.07644	0.08058	0.06673	0.04987	0.03559	0.02160	0.01772
Average value of expected portfolio return													
0.01436	0.01593	0.00973	0.00677	0.00596	0.00569	0.00513	0.00382	0.00403	0.00334	0.00249	0.00178	0.00108	0.00089
Maximal value of expected portfolio return													
0.04478	0.05628	0.03348	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772	0.01772
Minimal value of expected portfolio return													
0.00000	0.00146	0.00107	0.00141	0.00207	0.00200	0.00176	0.00094	0.00074	0.00049	0.00045	0.00021	0.00005	0.00000
CPU*													
3.635	5.039	5.195	8.736	7.364	6.459	6.676	6.443	6.271	6.146	7.067	6.146	6.443	7.894

*CPU seconds for proving optimality on a PC Intel® Core 2 Duo T9300, 2,5GHz, RAM 4GB /CPLEX 11

Figure 9.3: The example of solution results for the multi-period portfolio (model **M22**) with $1 - \alpha = 0.10$

VaR	**-10.00**	**-7.50**	**-5.00**	**-2.50**	**-2.00**	**-1.75**	**-1.50**	**-1.25**	**-1.00**	**-0.75**	**-0.50**	**-0.25**	**-0.05**	**0.00**
historical multi-period interval *k*		**Number of securities in each portfolio**												
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	3	5	6	4	7	4	6	6	6	6	7	8	5	0
3	1	5	4	7	7	8	9	8	7	9	8	8	6	0
4	2	3	3	5	6	9	7	7	9	7	8	11	12	0
5	1	2	3	7	8	10	8	8	8	7	7	7	8	0
6	0	4	4	8	8	10	11	11	8	9	10	11	11	0
7	1	3	5	6	7	7	7	8	7	6	8	7	8	0
8	2	3	3	5	7	10	5	6	10	10	8	13	13	0
9	2	6	6	6	11	5	6	8	10	8	12	14	9	0
10	2	3	4	5	11	14	15	9	19	13	19	12	16	0
11	2	1	6	9	13	14	16	16	11	20	18	17	17	0
12	5	3	5	7	14	13	14	11	16	12	10	12	14	0
13	5	5	9	11	16	13	16	6	15	20	12	12	12	0
14	3	2	4	7	10	6	6	9	6	4	6	11	4	0
15	2	2	5	5	10	10	12	9	16	14	14	8	15	0
16	3	2	4	8	7	8	14	17	14	17	12	15	15	0
17	2	5	6	8	9	8	6	4	9	5	9	6	6	0
18	3	4	6	8	10	7	8	8	12	7	9	9	10	0
19	3	3	3	4	6	7	10	9	11	7	13	11	8	0
20	2	4	5	10	10	12	11	8	7	8	8	8	9	0

Average number of securities													
-10.00	**-7.50**	**-5.00**	**-2.50**	**-2.00**	**-1.75**	**-1.50**	**-1.25**	**-1.00**	**-0.75**	**-0.50**	**-0.25**	**-0.05**	**0.00**
2	3	5	7	9	9	9	8	10	10	10	10	10	0
Maximal number of securities													
5	6	9	11	16	14	16	17	19	20	19	17	17	1
Minimal number of securities													
0	1	1	1	1	1	1	1	1	1	1	1	1	0

Figure 9.4: Number of securities in portfolios for each historical multi-period interval k with $1 - \alpha = 0.10$

9.7 shows the results for single-period portfolio optimization.

Expected portfolio return for whole investment period $\sum_{k=1}^{t}\left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_j^k\right)$													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.88354	0.88649	0.88943	0.89238	0.89297	0.89326	0.89356	0.89385	0.89415	0.89444	0.89474	0.89503	0.89527	0.89533
Average value of expected portfolio return													
0.04418	0.04432	0.04447	0.04462	0.04465	0.04466	0.04468	0.04469	0.04471	0.04472	0.04474	0.04475	0.04476	0.04477
Maximal value of expected portfolio return													
0.08833	0.08858	0.08884	0.08910	0.08915	0.08917	0.08920	0.08923	0.08925	0.08928	0.08930	0.08933	0.08935	0.08935
Minimal value of expected portfolio return													
0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553	0.01553
CPU*													
4.525	6.768	8.875	9.624	10.546	9.736	9.716	9.626	9.718	9.656	9.452	9.422	9.344	8.722

*CPU seconds for proving optimality on a PC Intel® Core 2 Duo T9300, 2,5GHz, RAM 4GB /CPLEX 11

Figure 9.5: The example of solution results for the multi-period portfolio with $1-\alpha=0.5$

Average number of securities													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
1	1	1	1	1	1	1	1	1	1	1	1	1	1
Maximal number of securities													
2	2	2	2	2	2	2	2	2	2	2	2	2	2
Minimal number of securities													
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Figure 9.6: Number of securities in portfolios for each historical multi-period interval k with $1-\alpha=0.5$

Figures 9.8–9.11 present example of the solution results for bi-objective multi-period portfolio **M23** - weighting approach for $VaR=-2.50$.

In this section numerical examples and some computational results are presented to illustrate possible applications of the proposed formulations of this multi-period optimization model **M23**. The examples are modeled on a real data form the Warsaw Stock Exchange.

Suppose that n=241 securities with historical quotations in t=20 investment periods, each of h=201 days, in total 4020 samples.

$1-\alpha$	Portfolio Return	Number of assets	MIP simplex iteration	B-&-B	CPU*
0.01	0.0170972	1	1374512	318186	839.44
0.05	0.0305677	3	1514015	250899	1909.19
0.10	0.0474076	3	3085511	477413	5521.30
0.20	0.0795746	3	358214	31771	259.59
0.50	0.0831270	2	1026	0	10.75
VaR=-2.00		r^{Min}=-100		m=4020	

*CPU seconds for proving optimality on a PC Intel® Core 2 Duo T9300, 2,5GHz, RAM 4GB /CPLEX 11

Figure 9.7: Solution results for the single-period portfolio with $1 - \alpha \in \{0.01; 0.05; 0.10; 0.20; 0.50\}$, $h = 4020$, $t = 1$

Security (Stock)	successive investment period *k*					
	1	2	3	4	5	6
ALMA						0.29
BUDIMEX				0.41	0.01	
DEBICA			0.98	0.02	0.02	0.03
ECHO					0.95	0.03
IRENA		0.99	0.01			
JUTRZENKA				0.56	0.01	0.01
KABLE						0.26
OPTIMUS					0.01	0.01
PROCHNIK	0.25	0.01				0.16
RAFAKO			0.01	0.01		
ZYWIEC						0.22
Expected Portfolios Return	1	2	3	4	5	6
	0.00443	0.07076	0.0243	0.02644	0.05499	0.01384
1-α	0.00025	0.01393	0.00771	0.00448	0.00945	0.00348

Figure 9.8: The solution results for bi-objective multi-period portfolio **M23** - weighting approach $\lambda = 0.5$, for successive investment period $k = 1, 2, 3, 4, 5, 6$

Security (Stock)	successive investment period k					
	7	8	9	10	11	12
13FORTUNA					0.02	
APATOR					0.01	0.85
FORTISPL		0.41	0.01			
IGROUP			0.98			
IRENA				0.80		
JUTRZENKA		0.03				
KOMPAP					0.02	
KOPEX					0.88	
KREDYTB	0.07	0.01				
LZPS		0.27				
MILLENNIUM		0.26	0.01			
NOVITA				0.14		
POLNA					0.01	
SUWARY						0.15
TIM				0.06		
WANDALEX					0.05	
ZYWIEC	0.93	0.02				
Expected Portfolios	7	8	9	10	11	12
Return	0.01994	0.02991	0.04735	0.02704	0.02705	0.02563
1-α	0.00672	0.00373	0.01119	0.00398	0.00672	0.00224

Figure 9.9: The solution results for bi-objective multi-period portfolio **M23** - weighting approach $\lambda = 0.5$, for successive investment period $k = 7, 8, 9, 10, 11, 12$

Security (Stock)	13	14	15	16	17	18
ALCHEMIA		0.03	0.61	0.01	0.01	0.01
APATOR	0.01	0.01	0.01	0.01	0.01	
ENERGOPOL					0.01	0.12
GANT				0.97	0.01	0.01
IBSYSTEM		0.36				
INDYKPOL		0.02				
KOPEX	0.98	0.01				
MASTERS					0.44	
MENNICA			0.013			
POLNORD						0.71
PONARFEH						0.13
PPWK			0.01			
SKOTAN					0.51	
SUWARY	0.01	0.01				
TUEUROPA			0.36	0.01	0.01	0.01
YAWAL		0.56				
Expected Portfolios	13	14	15	16	17	18
Return	0.05553	0.05832	0.07743	0.0818	0.08419	0.07558
1-α	0.00522	0.00746	0.00821	0.00995	0.00821	0.00299

Figure 9.10: The solution results for bi-objective multi-period portfolio **M23** - weighting approach $\lambda = 0.5$, for successive investment period $k = 13, 14, 15, 16, 17, 18$

The eighteen years horizon from 30th Jan 1991 to 30th Jan 2009 - consist of m=4020 historic daily quotations divided into t=20 investment periods (h=201 daily quotations each), with the selection of n=241 input securities for portfolio, quoted each day in the historical horizon. Probability of realization for expected securities returns is the same for each day and summed up for whole period to one.

The computational experiments have been performed using AMPL programming language and the CPLEX v.11 solver (with the default settings) on a laptop with Intel Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM. Computational time range is from a few seconds to minutes.

Figure 9.12 presents the comparison of computational time range for all multi-period portfolios - model **M23** - weighting approach.

Figures 9.13–9.16 presents results for model **M23** formulated by Reference Point Approach.

Security (Stock)	19	20
13FORTUNA	0.01	
ASTARTA	0.92	
ATLANTIS	0.02	
HOOP		0.31
HTLSTREFA		0.57
MOSTALPLC	0.02	
MUZA		0.09
OPTIMUS	0.02	
TUEUROPA	0.01	0.02
Expected Portfolios Return	19	20
	0.02371	0.01586
1-α	0.00771	0.00274

Figure 9.11: The solution results for bi-objective multi-period portfolio **M23** - weighting approach $\lambda = 0.5$, for successive investment period $k = 19, 20$

λ	1-λ	MIP simplex iteration	B-&-B	CPU*
0.5	0.5	4699533	357477	26544.8

*CPU seconds for proving optimality on a laptop with Intel® Core 2 Duo T9300 processor running at 2.5 GHz and with 4 GB RAM /CPLEX v.11

Figure 9.12: Computational time range

The accepted number of securities in portfolio is at least one security in each successive investment period. The basic parameters for the reference point method take on the following values: $f_1^{opt} = 1$, $f_2^{opt} = 0.05$, $\lambda = 0.5$, $\gamma = 0.01$.

VaR	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical successive investment period *k*	**Computational results for objective function (1-α) - probability that return of investment is not less than VaR in successive investment period k**													
1	0.000249	0.008955	0.011940	0.014179	0.015920	0.015920	0.016169	0.016915	0.017413	0.017413	0.017413	0.017413	0.017413	0.017413
2	0.000498	0.007960	0.010199	0.011940	0.012687	0.013433	0.014428	0.014925	0.015920	0.016169	0.016418	0.017413	0.017910	0.017910
3	0.000000	0.000995	0.002736	0.007214	0.008458	0.008706	0.008955	0.009204	0.010199	0.011443	0.012189	0.022886	0.021144	0.023632
4	0.000000	0.000000	0.000000	0.001741	0.003483	0.004229	0.005721	0.008458	0.008955	0.008458	0.011940	0.013433	0.015423	0.021891
5	0.000000	0.002736	0.003731	0.009453	0.010697	0.012189	0.013682	0.015174	0.015423	0.011194	0.012935	0.019154	0.020149	0.016667
6	0.000000	0.000249	0.000249	0.002239	0.002488	0.004478	0.006965	0.006716	0.007463	0.010697	0.013682	0.017910	0.020149	0.024627
7	0.000249	0.001244	0.002239	0.004726	0.005473	0.007214	0.007463	0.010199	0.013184	0.015920	0.019403	0.025373	0.022139	0.022637
8	0.000000	0.000000	0.000000	0.003234	0.003483	0.003483	0.004726	0.003980	0.006468	0.006965	0.009204	0.017910	0.014925	0.023881
9	0.000000	0.000249	0.000746	0.008955	0.011940	0.012438	0.013930	0.012935	0.015423	0.015672	0.016667	0.017910	0.013930	0.022637
10	0.000000	0.000000	0.000000	0.001493	0.002985	0.004975	0.004229	0.005721	0.005473	0.009701	0.010199	0.013930	0.012687	0.017164
11	0.000000	0.000249	0.000746	0.004229	0.002736	0.006716	0.002985	0.001990	0.009701	0.010448	0.012189	0.015174	0.016667	0.021642
12	0.000000	0.000249	0.000000	0.000746	0.001990	0.001741	0.000995	0.001990	0.004229	0.006468	0.009453	0.008209	0.015920	0.021642
13	0.000498	0.000746	0.001493	0.002736	0.003980	0.004975	0.005721	0.006468	0.007711	0.008706	0.010945	0.014179	0.011443	0.018657
14	0.000000	0.000000	0.000249	0.004478	0.002239	0.005473	0.005970	0.009950	0.008706	0.009950	0.012687	0.008209	0.013184	0.021891
15	0.000498	0.002239	0.004229	0.007463	0.009701	0.009950	0.011940	0.011194	0.012935	0.014179	0.017164	0.008706	0.020149	0.012438
16	0.000000	0.000995	0.002736	0.009701	0.010697	0.011194	0.013682	0.014677	0.014925	0.016169	0.018159	0.012687	0.021891	0.019403
17	0.000249	0.000498	0.002488	0.008458	0.009701	0.009453	0.011692	0.014428	0.015672	0.016667	0.017413	0.007214	0.008955	0.021393
18	0.000000	0.000000	0.000995	0.002985	0.003483	0.003731	0.005473	0.006965	0.006965	0.008706	0.009950	0.005473	0.007214	0.025373
19	0.000249	0.000249	0.000995	0.002239	0.002239	0.004726	0.005970	0.006219	0.007214	0.007711	0.011692	0.014179	0.022139	0.022886
20	0.000000	0.000000	0.000498	0.002488	0.002736	0.003980	0.004726	0.006468	0.007960	0.009453	0.011940	0.014677	0.016667	0.018159

Average value													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.00012	0.00138	0.00231	0.00553	0.00636	0.00745	0.00827	0.00923	0.01060	0.01160	0.01358	0.01460	0.01650	0.02060
Maximal value													
0.00050	0.00896	0.01194	0.01418	0.01592	0.01592	0.01617	0.01692	0.01741	0.01741	0.01940	0.02537	0.02214	0.02537
Minimal value													
0.00000	0.00000	0.00000	0.00075	0.00199	0.00174	0.00100	0.00199	0.00423	0.00647	0.00920	0.00547	0.00721	0.01244

Figure 9.13: The solution results for objective function $1 - \alpha$

Figure 9.13 presents the solution results (model **M23** formulated by Reference Point Approach) for objective function $1 - \alpha$ probability that return of investment is not less than in each successive investment period k.

Figure 9.14 presents the solution results (model **M23** formulated by Reference Point Approach) for the objective function of expected portfolios return in each successive investment period k.

Figure 9.15 presents number of securities in the computed portfolios (model **M23** formulated by Reference Point Approach) for each successive investment period k.

Figure 9.16 presents computational time range and the solution values (model **M23** formulated by Reference Point Approach) of γ - the deviation from the reference solution.

VaR	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
historical successive investment period k	**Computational results for objective function** $\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk}$ **expected portfolios return**													
1	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724	0.017724
2	0.070761	0.070761	0.070761	0.067763	0.067635	0.067371	0.067544	0.070448	0.069815	0.067240	0.067694	0.030125	0.028664	0.028664
3	0.024444	0.024444	0.024444	0.023322	0.024444	0.024444	0.024034	0.024444	0.023868	0.022673	0.023463	0.002359	0.005598	0.002089
4	0.027760	0.026885	0.025954	0.022859	0.023592	0.022952	0.023908	0.026289	0.025925	0.019273	0.022895	0.007076	0.017101	0.010134
5	0.055988	0.055465	0.055966	0.055219	0.053645	0.054111	0.053835	0.055465	0.053127	0.040806	0.040221	0.023383	0.011053	0.022263
6	0.016764	0.015921	0.014679	0.012419	0.010556	0.011936	0.013085	0.012418	0.010993	0.012238	0.011124	0.004557	0.000996	0.001355
7	0.020589	0.020589	0.019941	0.016857	0.016613	0.017639	0.015481	0.015616	0.016691	0.015112	0.002549	0.012430	0.011628	0.003331
8	0.031882	0.031354	0.030397	0.029042	0.027955	0.027076	0.026588	0.025508	0.026435	0.026012	0.025394	0.011218	0.019939	0.006816
9	0.048203	0.043892	0.039017	0.043258	0.045281	0.044065	0.044344	0.040406	0.045859	0.044854	0.045469	0.048014	0.013503	0.006938
10	0.028001	0.027716	0.026827	0.024002	0.025275	0.025197	0.022398	0.022944	0.022575	0.022790	0.018884	0.007076	0.003166	0.008150
11	0.025639	0.026031	0.026184	0.024804	0.020658	0.024065	0.018287	0.016384	0.022287	0.021258	0.021400	0.011683	0.000835	0.005386
12	0.025682	0.025682	0.025548	0.024555	0.024519	0.024356	0.021084	0.021295	0.023018	0.019324	0.020103	0.002275	0.005586	0.004849
13	0.056018	0.056018	0.055462	0.053231	0.052851	0.052570	0.052162	0.051774	0.050749	0.046546	0.047800	0.028891	0.017758	0.008715
14	0.058699	0.056988	0.055645	0.055334	0.052905	0.053225	0.053446	0.056469	0.052085	0.051592	0.051437	0.036650	0.025911	0.023462
15	0.071572	0.073183	0.073669	0.075568	0.076221	0.076269	0.076789	0.073240	0.076417	0.073869	0.076417	0.008486	0.008165	0.009565
16	0.080490	0.081758	0.080721	0.080926	0.078377	0.077874	0.080885	0.080420	0.078805	0.075332	0.077503	0.009849	0.024172	0.015486
17	0.085573	0.085815	0.085058	0.084567	0.082168	0.079443	0.082134	0.083629	0.082977	0.081912	0.080363	0.049416	0.039606	0.008291
18	0.076582	0.075889	0.075833	0.075581	0.073863	0.072654	0.074755	0.073789	0.072593	0.069594	0.069667	0.049453	0.050654	0.050313
19	0.024858	0.024246	0.023692	0.018291	0.015745	0.017869	0.018093	0.016482	0.015297	0.012908	0.014121	0.010479	0.007908	0.024858
20	0.017350	0.016878	0.017081	0.015555	0.013707	0.014389	0.012835	0.013079	0.013483	0.009589	0.010298	0.014722	0.001857	0.003602

Expected portfolio return for whole investment period $\sum_{k=1}^{t}\left(\sum_{i=(k-1)h+1}^{kh} p_i \sum_{j=1}^{n} r_{ij} x_{jk}\right)$													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
0.86458	0.85724	0.84460	0.82088	0.80374	0.80523	0.79941	0.79782	0.80072	0.75064	0.74452	0.38587	0.31182	0.26199
Average value of expected portfolio return													
0.04323	0.04286	0.04223	0.04104	0.04019	0.04026	0.03997	0.03989	0.04004	0.03753	0.03723	0.01929	0.01559	0.01310
Maximal value of expected portfolio return													
0.08557	0.08581	0.08506	0.08457	0.08217	0.07944	0.08213	0.08363	0.08298	0.08191	0.08036	0.04945	0.05065	0.05031
Minimal value of expected portfolio return													
0.01676	0.01592	0.01468	0.01242	0.01056	0.01194	0.01283	0.01242	0.01099	0.00959	0.00255	0.00227	0.00083	0.00135

Figure 9.14: The solution results for objective function - the expected portfolios return

VaR	-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
Historical successive investment period *k*		**Number of securities in each portfolio**												
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	5	3	4	4	3	3	6	7	5	1	1
3	2	2	2	4	2	2	3	2	4	5	6	5	1	1
4	2	3	2	5	3	5	6	3	5	8	8	6	1	1
5	3	4	3	5	5	7	8	4	7	10	12	6	3	1
6	3	6	7	12	13	14	12	9	11	11	11	7	2	1
7	1	1	2	5	4	4	6	5	4	5	4	1	1	1
8	2	4	5	10	10	9	13	13	12	16	17	5	14	1
9	1	6	7	9	7	9	8	12	7	10	8	2	22	1
10	1	2	4	6	6	6	9	10	12	11	10	14	2	1
11	2	3	4	6	10	6	12	10	11	15	16	13	4	1
12	1	1	2	4	4	5	6	8	7	10	11	8	4	1
13	2	2	3	5	7	9	8	9	10	13	18	7	14	1
14	5	6	9	10	15	17	15	12	15	20	20	22	18	1
15	4	5	5	9	9	12	8	12	12	17	14	24	10	1
16	6	4	5	6	7	9	6	5	7	11	10	14	3	1
17	6	6	8	6	10	12	8	9	9	11	13	22	24	1
18	6	6	7	7	9	12	10	10	14	17	18	35	31	3
19	1	4	2	11	10	9	9	13	15	20	15	19	3	1
20	1	2	2	5	6	7	8	11	8	11	12	5	2	1

Average number of securities													
-10.00	-7.50	-5.00	-2.50	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50	-0.25	-0.05	0.00
3	4	4	7	7	8	8	8	9	11	12	11	8	1
Maximal number of securities													
6	6	9	12	15	17	15	13	15	20	20	35	31	3
Minimal number of securities													
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Figure 9.15: Number of securities in computed portfolios for each historical successive investment period

VaR	δ	MIP simplex iterations	branch-and-bound nodes	CPU	GAP
-10.00	0.067711	15104	805	43.1	-
-7.50	0.071381	3182319	441196	10800.0	1.85%
-5.00	0.077699	3169107	195210	10800.1	8.72%
-2.50	0.089562	1668609	116278	10800.1	29.31%
-2.00	0.098133	1554694	76373	10800.2	32.57%
-1.75	0.097387	1177463	69993	10800.3	36.15%
-1.50	0.100295	1162975	65026	10800.3	38.61%
-1.25	0.101089	1713553	71822	10800.4	39.98%
-1.00	0.099639	2744317	95456	10800.1	42.44%
-0.75	0.124678	2730531	101598	10800.1	47.37%
-0.50	0.127738	2824989	84196	10800.4	51.10%
-0.25	0.307067	3022921	76426	10800.1	65.44%
-0.05	0.344089	3027166	71625	10800.1	67.75%
0.00	0.369005	3561349	84385	10800.0	71.55%

*CPU seconds for proving optimality on a laptop with Intel® Core 2 Duo T9300 processor running at 2.5 GHz and with 4 GB RAM /CPLEX v.11

Figure 9.16: Computational time range and the solution results for γ

Table 9.2: The solution results for the weighting approach (model **M26**) with 100 historical quotations

β_1	β_2	β_3	$1-\alpha$	Portfolio return	Amount of capital	Number of assets	MIP simplex iteration	B-&-B nodes	CPU / GAP
0.80	0.10	0.10	0.000	0.658086	1	7	1061	101	4.11%
0.10	0.80	0.10	0.280	1.738270	1	1	56	0	0.17%
0.10	0.10	0.80	0.270	1.731680	1	2	112	17	0.82%
0.70	0.15	0.15	0.040	0.861917	1	13	25788	3910	65.30%
0.15	0.70	0.15	0.280	1.738270	1	1	56	0	1.59%
0.15	0.15	0.70	0.270	1.731680	1	2	112	17	0.77%
0.60	0.20	0.20	0.230	1.633050	1	8	4741	1008	19.17%
0.20	0.60	0.20	0.280	1.738270	1	1	56	0	0.16%
0.20	0.20	0.60	0.270	1.731680	1	2	112	17	0.71%
0.50	0.25	0.25	0.260	1.713360	1	4	712	124	4.12%
0.25	0.50	0.25	0.280	1.738270	1	1	63	0	0.22%
0.25	0.25	0.50	0.270	1.731680	1	2	112	17	0.71%
0.40	0.30	0.30	0.270	1.731680	1	2	188	37	1.92%
0.30	0.40	0.30	0.270	1.731680	1	2	76	2	0.50%
0.30	0.30	0.40	0.270	1.731680	1	2	112	17	0.88%

*CPU seconds for proving optimality on a PC Pentium III, RAM 512MB/CPLEX 9.1

Column "number of assets" defines amount of stocks in optimal solutions.

In the tables, column "MIP simplex iteration" shows the number of mixed integer programming simplex iterations until the solution is presented.

Column "B-&-B nodes" shows the number of searched nodes in the branch and bound tree until the solution presented.

Column "GAP" shows percentage difference between obtained solution and the best LP-relaxation based bound calculated by the CPLEX solver.

Table 9.2 presents solution results for the weighting approach (model **M26**) with 100 historical quotations.

Table 9.3 shows the solution results for maximization of expected portfolio return (model **M26**) with 500 historical quotations.

Table 9.4 presents the results for the maximization of expected portfolio return (model **M25**) with 1758 historical quotations.

Table 9.5 presents the solution results for the weighting approach (model **M25**) with 500 historical quotations.

Table 9.6 shows the solution results for the weighting approach (model **M25**) with 1758 historical quotations.

Table 9.7 presents the number of assets in optimal portfolio for lexicographic approach (model **M12**) with 500 historical quotations.

Table 9.8 shows the number of assets in optimal portfolio for lexicographic approach (model **M12**) with 1758 historical quotations.

Table 9.8 presents examples of CPU time for computational experiments for optimal portfolio for lexicographic approach (model **M12**) with 1758 historical quotations.

The computational time for the optimization model with objective function (model **M12**) requires several CPU minutes for finding the first feasible solution.

The total computational time ranges from a few seconds to minutes or even hours depending on the number of historical quotations in the optimization problem.

Table 9.3: The solution results for the weighting approach (model **M26**) with 500 historical quotations

β_1	β_2	β_3	$1-\alpha$	Portfolio return	Amount of capital	Number of assets	MIP simplex iteration	B-&-B nodes	CPU / GAP
0.80	0.10	0.10	0.018	0.473987	1	35	190977472	7709994	71271.21
0.10	0.80	0.10	0.360	1.295300	1	1	361	1	0.51%
0.10	0.10	0.80	0.338	1.280820	1	6	7896177	1660831	4280.88
0.70	0.15	0.15	0.042	0.600772	1	34	62253914	5882993	22595.40
0.15	0.70	0.15	0.360	1.295300	1	1	766	21	1.00%
0.15	0.15	0.70	0.340	1.281930	1	5	11217491	2391321	5431.07
0.60	0.20	0.20	0.062	0.666516	1	34	30208779	5402504	12736.50
0.20	0.60	0.20	0.360	1.295300	1	1	3419	901	4.72%
0.20	0.20	0.60	0.338	1.280660	1	6	6192835	1402627	3520.78
0.50	0.25	0.25	0.274	1.186270	1	10	23069048	5320236	39033.00
0.25	0.50	0.25	0.358	1.294360	1	2	114568	61391	164.66
0.25	0.25	0.50	0.334	1.277550	1	6	41459235	10512354	26839.90
0.40	0.30	0.30	0.332	1.273370	1	9	29281271	5992131	13337.90
0.30	0.40	0.30	0.340	1.282980	1	4	2047901	560025	1731.51
0.30	0.30	0.40	0.336	1.278810	1	6	49893067	11875407	30246.80
0.34	0.33	0.33	0.336	1.278810	1	6	36991485	7984601	20446.20

*CPU seconds for proving optimality on a PC Pentium III, RAM 512MB/CPLEX 9.1

Table 9.4: The results for the maximization of expected portfolio return (model **M25**) for 1758 quotations

$1-\alpha$	VaR	Portfolio return	Amount of capital	Number of assets	MIP simplex iteration	B-&-B nodes	GAP	CPU
0.10	-2.00	0.406521	1.0000	16	22203	2601	10.24%	3671.41
0.15	-2.00	0.450744	1.0000	8	43873	9401	0.66%	3599.88
0.15	-1.00	0.357077	1.0000	33	17534	1101	7.14%	3600.14
0.15	-0.50	0.223021	0.8436	61	105900	2601	99.43%	32182.40
0.50	-0.25	0.109703	0.3421	28	3813	100	319.88%	1176.50

*CPU seconds for proving optimality on PC Pentium III, RAM 512MB /CPLEX 9.1

The selected results obtained for models **M7**, **M1** are presented in table 9.11 and in figures 9.17–9.24.

Table 9.11 shows the optimal values of $CVaR$, VaR, and the expected portfolio return for different confidence level α and the size of the input data set. In all cases the CPU time increases when the confidence level decreases. The number of securities in the optimal portfolios varies between 14 and 39 assets. The relation between the conditional value-at-risk $CVaR$ and the confidence level α is also shown in figure . VaR and $CVaR$ increase as the confidence level decreases.

Figure 9. shows the number of securities in the computed portfolio and the computational time range for different size of input data set and model **M1**. The number of stocks selected for the optimal portfolio is independent on the confidence level and the size of input data set.

The computational results for model **M1** are presented in figures 9.17–9.4 and tables 9.11–9.12.

The conditional value-at-risk is more negative than value-at-risk, which is clearly shown in Figures 9.17, 9.19 and 9.20.

The confidence level α has a strong impact on obtained values $CVaR$ and VaR.

Table 9.5: The results for the maximization of expected portfolio return (model **M25**) for 500 quotations

$1-\alpha$	VaR	Portfolio return	Amount of capital	Number of assets	MIP simplex iteration	B-&-B nodes	GAP/CPU
0.01	-10.00	1.221675	1.0000	3	47	6	3.35
0.01	-5.00	0.937749	1.0000	10	1044	91	19.72
0.01	-4.00	0.849007	1.0000	10	2588	192	31.42
0.01	-3.00	0.743942	1.0000	13	12139	512	84.64
0.01	-2.00	0.629044	1.0000	17	33359	1105	242.11
0.01	-1.50	0.532917	1.0000	23	111944	2498	544.31
0.01	-1.00	0.396546	1.0000	30	297576	5462	1689.62
0.01	-0.50	0.198811	0.5314	31	826053	11333	5295.86
0.01	-0.25	0.099405	0.2657	31	671318	8527	4690.36
0.05	-10.00	1.295303	1.0000	1	34	0	0.88
0.05	-5.00	1.225334	1.0000	5	273	22	7.47
0.05	-4.00	1.141190	1.0000	9	11445	2450	161.97
0.05	-3.00	1.020270	1.0000	13	334933	33001	2.84%
0.05	-2.00	0.883416	1.0000	19	86761	3401	16.13%
0.10	-3.00	1.160440	1.0000	10	53836	6801	2.87%
0.15	-2.00	1.118940	1.0000	14	32074	3790	7.73%
0.20	-2.00	1.221670	1.0000	7	39648	7501	1.87%
0.25	-1.50	1.234330	1.0000	8	38871	8801	1.56%

Table 9.6: The solution results for the weighting approach (model **M25**) with 1758 historical quotations

β_1	β_2	VaR	f_2^{opt}	$1-\alpha$	Portfolio return	Amount of capital	Number of assets	MIP simplex iteration	CPU
0.9	0.1	-2.00	0.40	0.111490	0.400819	1	4	222	4.78
0.5	0.5	-2.00	0.40	0.112059	0.400921	1	4	222	4.95
0.1	0.9	-2.00	0.40	0.114334	0.400459	1	4	224	4.73
0.9	0.1	-2.00	0.45	0.151308	0.450515	1	3	294	5.33
0.5	0.5	-2.00	0.45	0.150171	0.450000	1	3	294	5.06
0.1	0.9	-2.00	0.45	0.150171	0.450000	1	3	293	4.73
0.9	0.1	-1.00	0.34	0.134243	0.340825	1	18	615	10.10
0.5	0.5	-1.00	0.34	0.135381	0.340647	1	18	671	10.00
0.1	0.9	-1.00	0.34	0.133675	0.340722	1	18	721	12.03
0.9	0.1	-0.50	0.20	0.110353	0.202772	1	56	1040	30.81
0.5	0.5	-0.50	0.20	0.111490	0.204490	1	56	1123	34.82
0.1	0.9	-0.50	0.20	0.110353	0.202382	1	56	1185	37.07
0.9	0.1	-0.25	0.10	0.192833	0.141738	1	66	1748	49.60
0.5	0.5	-0.25	0.10	0.191126	0.142552	1	68	1733	49.98
0.1	0.9	-0.25	0.10	0.191695	0.142652	1	68	1559	57.72

Table 9.7: The solution results for the weighting approach (model **M27**) with 500 historical quotations

VaR	f_2^{opt}	$1-\alpha$	Portfolio return	Amount of capital	Numer of assets	MIP simplex iteration	CPU
-10.00	1.22168	0.010	1.221670	1	3	14	2.31
-5.00	0.94774	0.016	0.937749	1	6	124	1.04
-4.00	0.84900	0.014	0.849007	1	9	180	1.32
-3.00	0.74394	0.016	0.752234	1	12	213	3.35
-2.00	0.62900	0.016	0.629000	1	12	254	3.79
-1.50	0.53300	0.020	0.533000	1	21	198	3.74
-1.00	0.39560	0.022	0.421036	1	21	295	3.46
-0.50	0.19900	0.050	0.219878	1	30	392	3.46
-0.25	0.09940	0.104	0.232379	1	41	499	4.78
-10.00	1.29530	0.022	1.295300	1	1	19	3.13
-5.00	1.22530	0.060	1.226390	1	2	39	1.53
-4.00	1.14000	0.060	1.140000	1	8	47	3.02
-3.00	1.02000	0.058	1.020000	1	8	54	0.88
-2.00	0.88000	0.070	0.880000	1	12	101	1.15
-3.00	1.16044	0.124	1.160440	1	7	86	0.88
-2.00	1.11894	0.160	1.122190	1	8	106	1.10
-2.00	1.22167	0.212	1.221670	1	5	119	0.88
-1.50	1.23433	0.264	1.234330	1	5	146	0.99

* $\lambda = 0.5$; Objective function: $1-\alpha$ and expected portfolio return

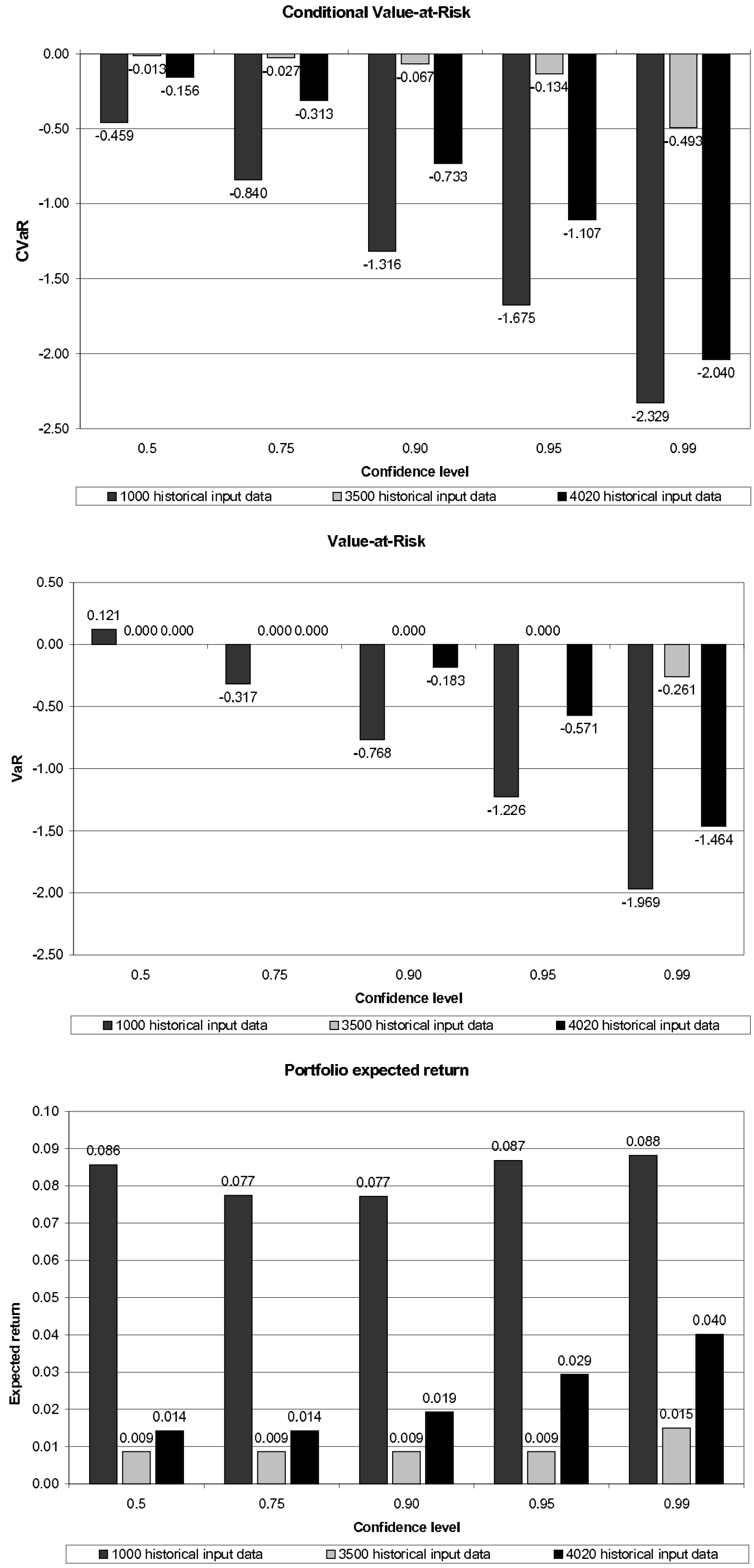

Figure 9.17: Comparison of $CVaR$, VaR and the expected return for model **M1** with 1000, 3500 and 4020 historical input data.

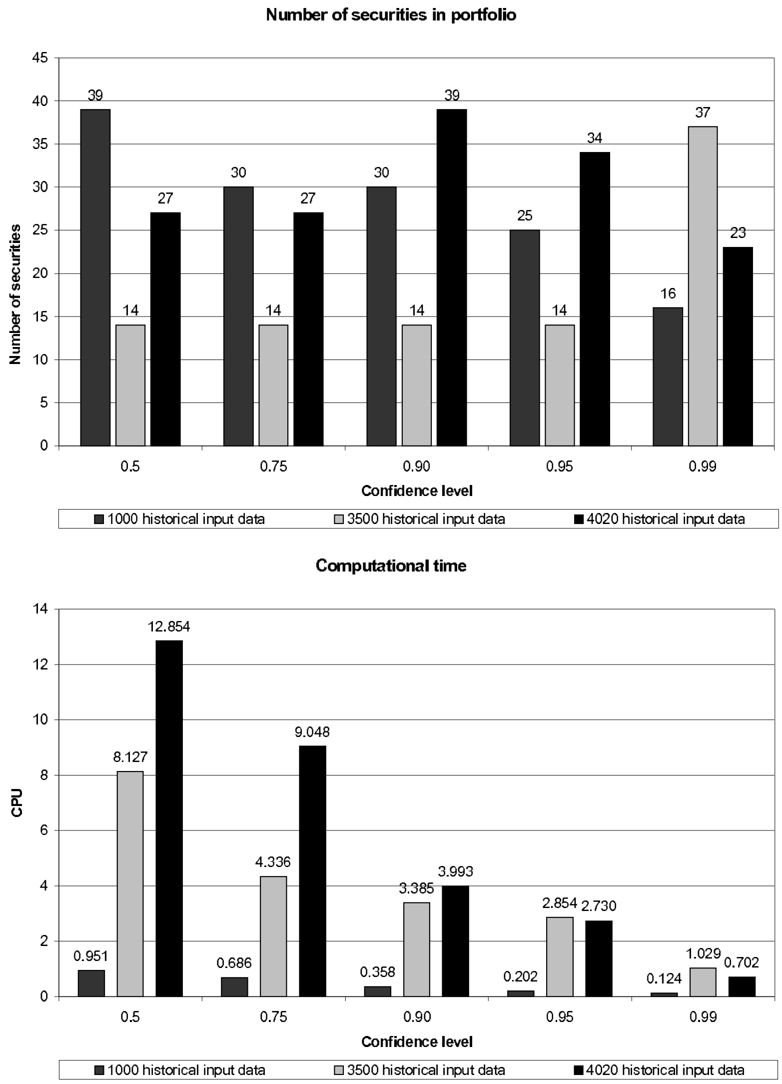

Figure 9.18: Number of securities in an optimal portfolio and computational time range for model **M1**.

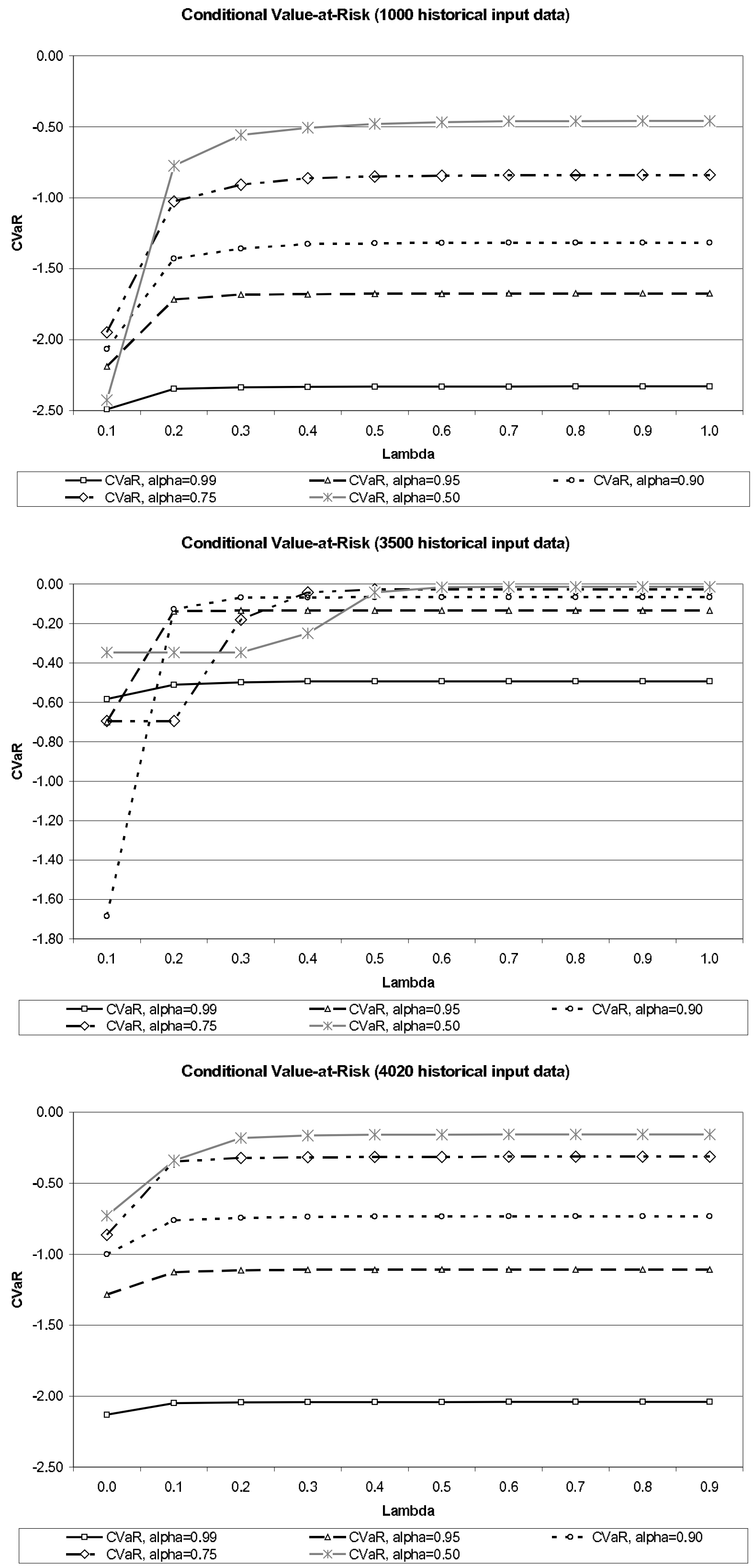

Figure 9.19: Conditional Value-at-Risk for different confidence levels - model **M1**

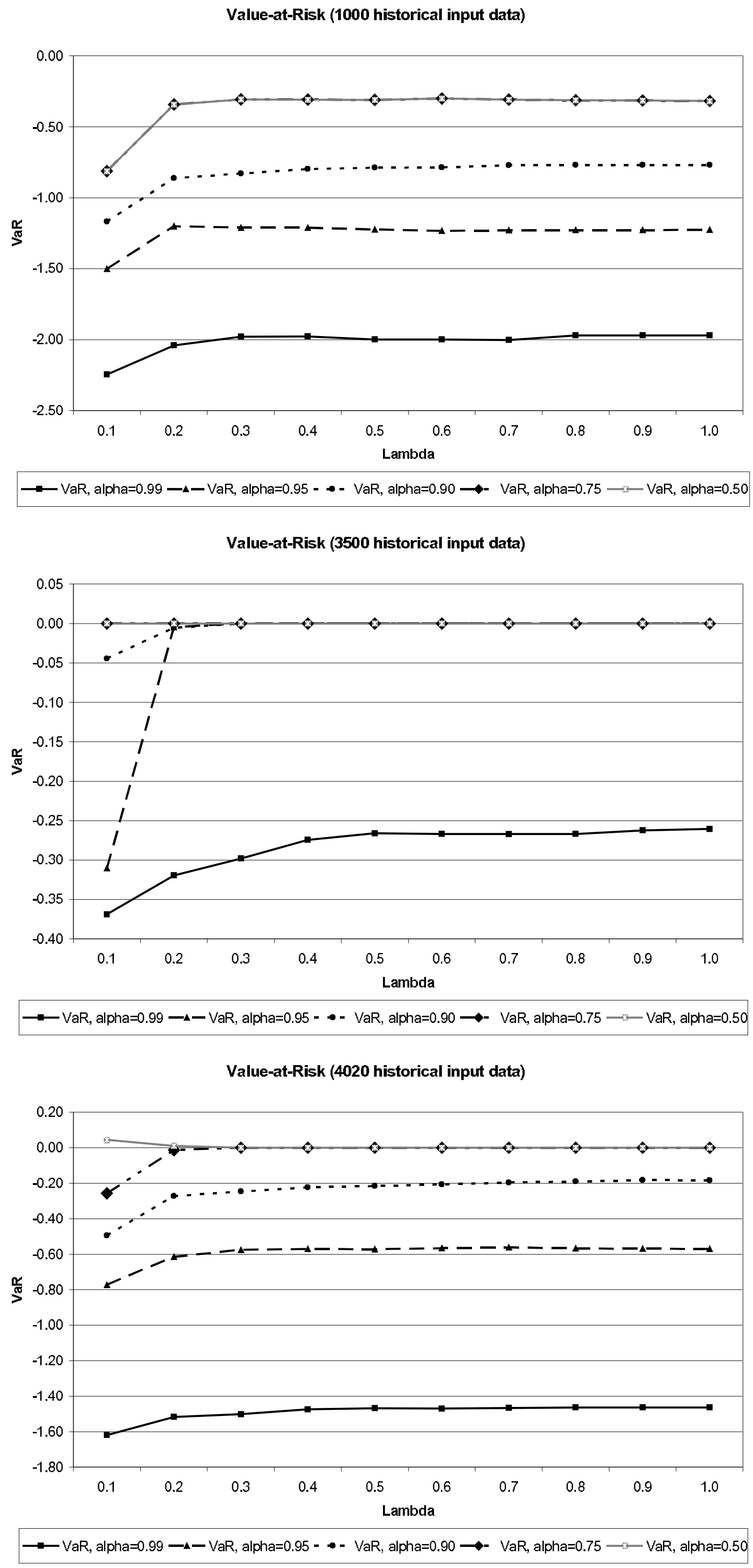

Figure 9.20: Value-at-Risk for different confidence levels - model **M1**

Table 9.8: Number of assets in optimal portfolio for lexicographic approach (model **M12**) with 1758 historical quotations

$1-\alpha$	VaR	f_2^{opt}	Number of assets	$sum_{j=1}^{n} z_j$
0.10	-2.00	0.40	16	4
0.15	-2.00	0.45	8	3
0.15	-1.00	0.34	33	18
0.15	-0.50	0.20	61	56
0.50	-0.25	0.10	68	68

The expected portfolio returns for different confidence level α and weight λ and for the three different historical portfolios are shown in Figure 9.17 and 9.21.

Figures 9.18 and 9.22 shows that the number of securities selected for the optimal portfolio does not clearly depend either on the confidence level α, or on the size of the historical input data set. In weighted-sum program the weight λ only slightly influences the number of selected stock.

Figure 9.23 presents the efficient frontiers of the bi-objective model **M1** - conditional value-at-risk vs. expected return for the three different historical scenarios. The trade-off between the conditional value-at-risk and expected portfolio return is clearly shown as a concave efficient frontier.

Figure 9.18 presents the CPU time required to obtain the optimal solution for **M1** model for different historical input data set.

The computational time increases with the confidence level α and the size of input data set.

The examples of nondominated solutions for model **M1** with different values of the confidence level *alpha* and for historic portfolios (scenarios) are presented in table 9.12.

Table 9.9: Number of assets in optimal portfolio for lexicographic approach (model **M12**) with 500 historical quotations

$1-\alpha$	VaR	f_2^{opt}	Number of assets z_j	$sum_{j=1}^{n} z_j$
0.01	-10.00	1.2217	3	3
0.01	-5.00	0.9377	10	6
0.01	-4.00	0.8490	10	9
0.01	-3.00	0.7439	13	12
0.01	-2.00	0.6290	17	12
0.01	-1.50	0.5330	23	21
0.01	-1.00	0.3956	30	21
0.01	-0.50	0.1990	31	30
0.01	-0.25	0.0994	31	31
0.05	-10.00	1.2953	1	1
0.05	-5.00	1.2253	5	2
0.05	-4.00	1.1400	9	8
0.05	-3.00	1.0200	13	8
0.05	-2.00	0.8800	19	12
0.10	-3.00	1.1604	10	7
0.15	-2.00	1.1189	14	8
0.20	-2.00	1.2217	7	5
0.25	-1.50	1.2343	8	5

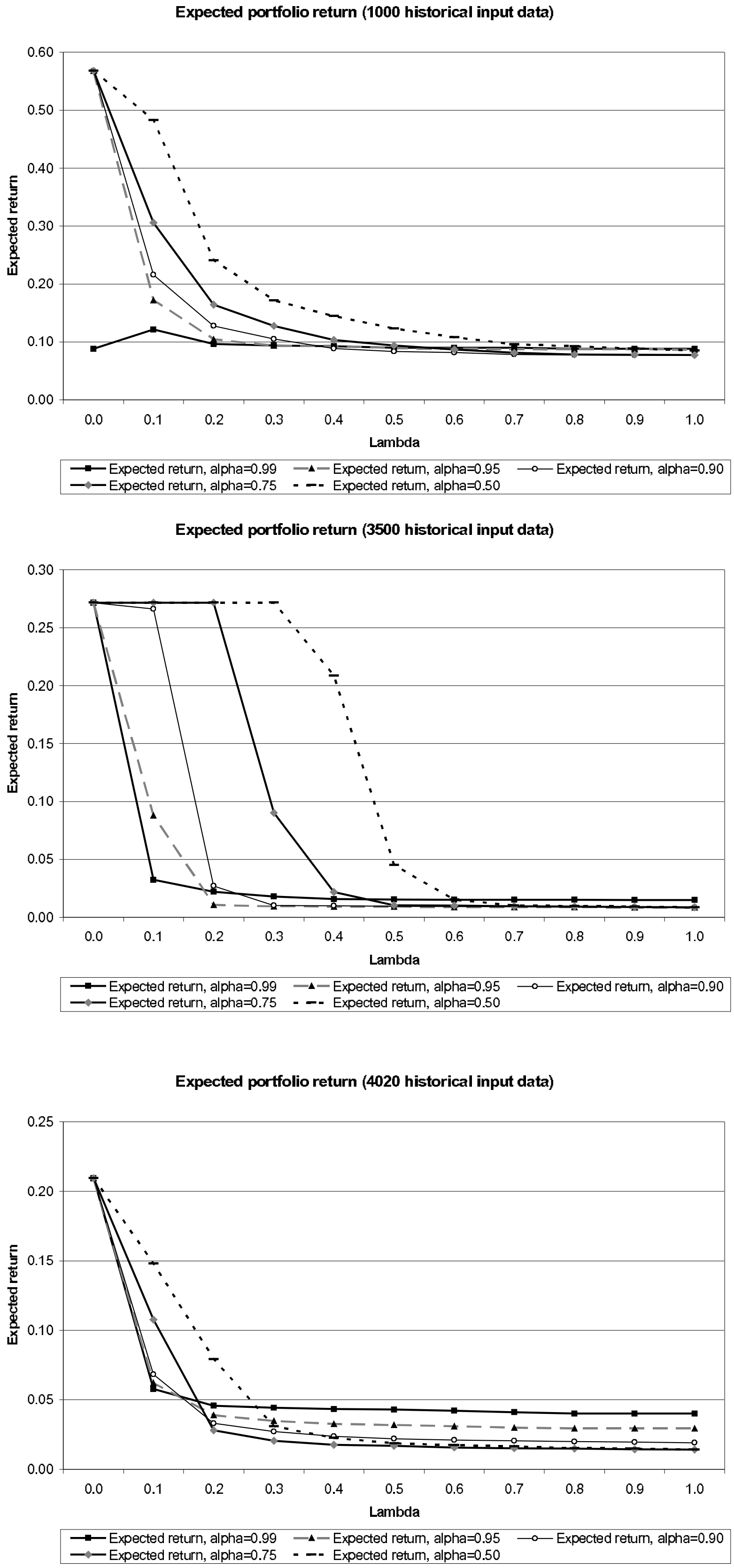

Figure 9.21: Expected portfolio return for different confidence levels - model **M1**

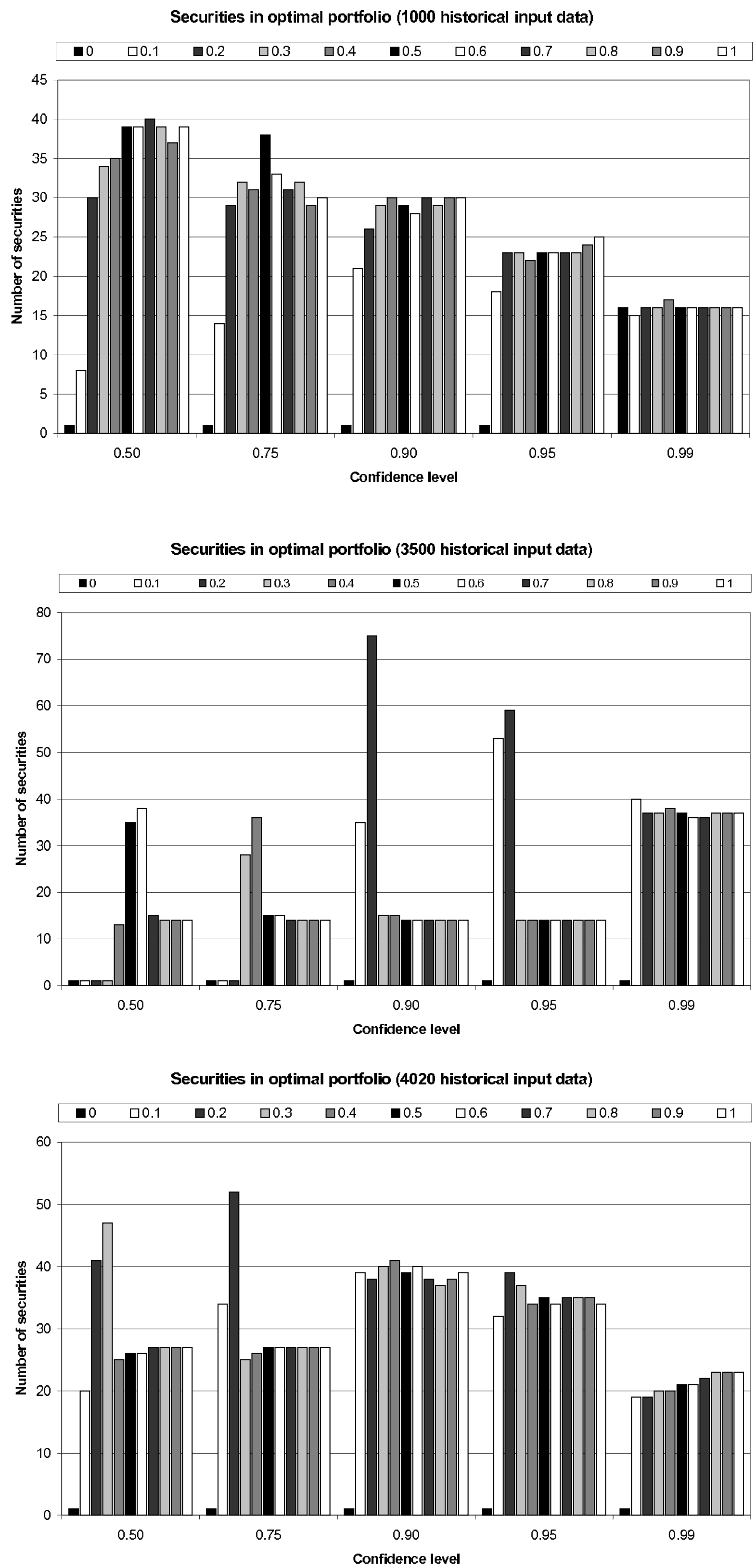

Figure 9.22: Number of assets in the optimal portfolio for 1000, 3500, 4020 historical input data - model **M1**

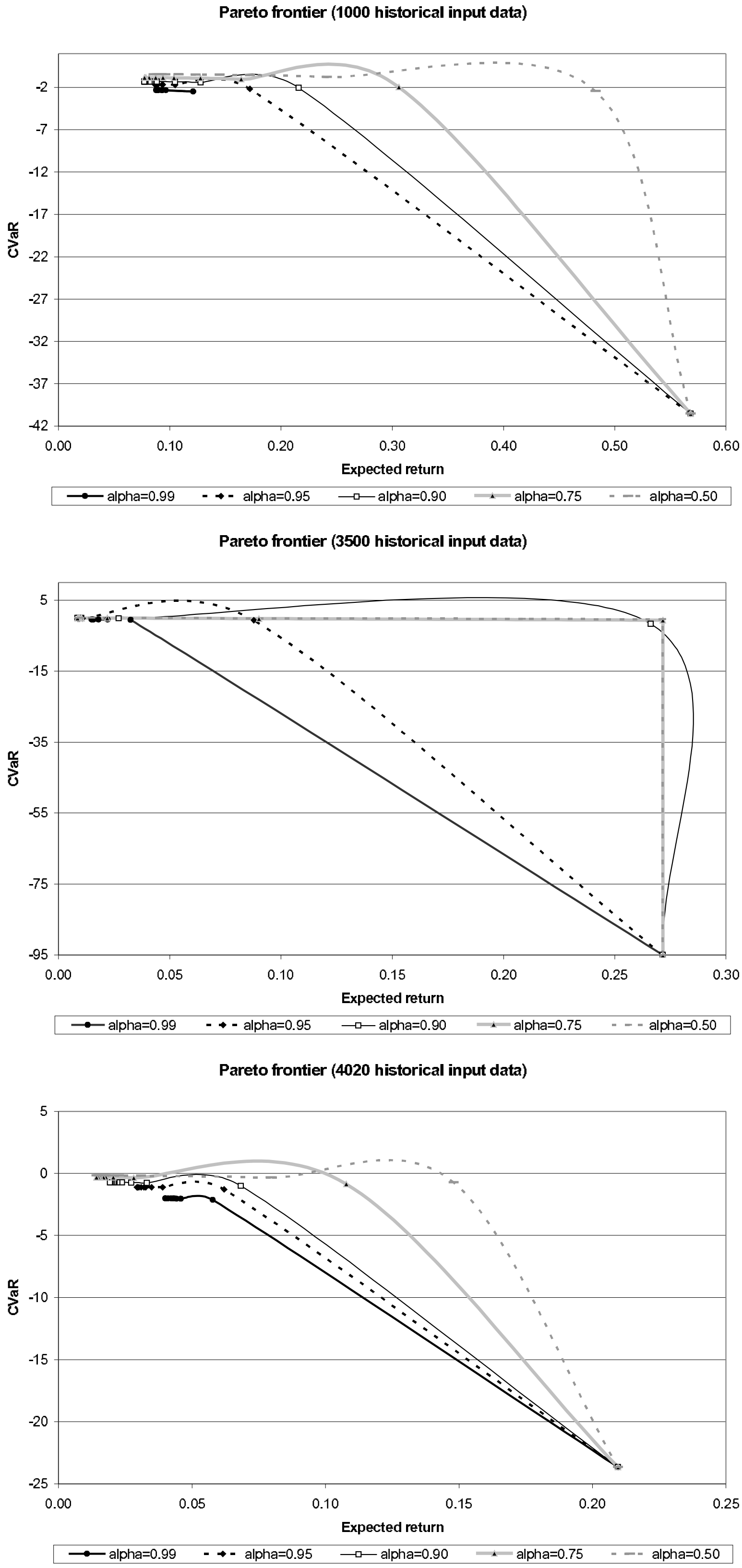

Figure 9.23: Pareto frontier for the bi-objective model **M1** for the three different scenario sizes.

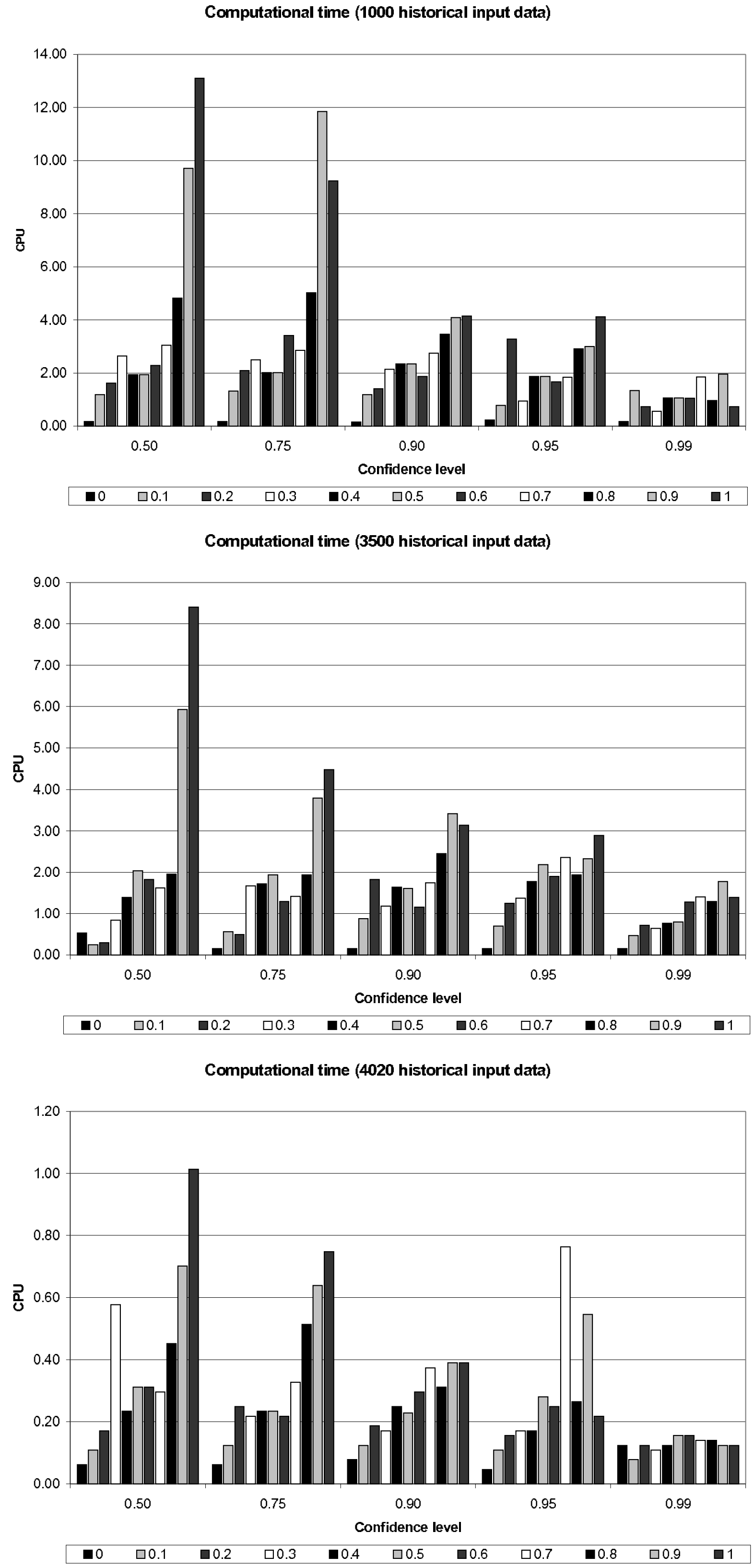

Figure 9.24: Central processing unit - computational time for model **M1** and the three different historical scenarios.

Table 9.10: Examples of CPU time for computational experiments for optimal portfolio for lexicographic approach (model **M12**) with historical quotations

$1-\alpha$	VaR	f_2^{opt}	MIP simplex iteration	B-&-B nodes	CPU
0.01	-10	1.2217	27	10	110.02
0.01	-5	0.9377	610	192	880.79
0.01	-4	0.8490	3217	1024	5045.62
0.01	-3	0.7439	7779	2300	10272.80
0.01	-2	0.6290	19980	4014	21677.90

Table 9.11: Solutions results for model **M7** and **M1** with 4020, 3500, 1000 historical input data.

Confidence level α	0.99	0.95	0.90	0.75	0.50
input data, Var.=1136, Cons.=1001, Nonz.=118100					
CVaR	-2.329320	-1.674800	-1.316210	-0.839905	-0.458828
VaR	-1.968690	-1.225590	-0.767942	-0.316816	0.120616
Expected portfolio return	0.088178	0.086781	0.077124	0.077459	0.085606
No. of securities in portfolio	16	25	30	30	39
Dual simplex iterations	226	506	948	1716	2202
CPU [(a)]	0.124	0.202	0.358	0.686	0.951
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.360628	0.449209	0.548270	0.523088	0.579444
No. of non-zero R_i	4	38	86	270	478
Confidence level α	0.99	0.95	0.90	0.75	0.50
input data, Var.=3736, Cons.=3501, Nonz.=332550					
CVaR	-0.492633	-0.133975	-0.066987	-0.026795	-0.013398
VaR	-0.260505	0.000000	0.000000	0.000000	0.000000
Expected portfolio return	0.014985	0.008625	0.008625	0.008625	0.008625
No. of securities in portfolio	37	14	14	14	14
Dual simplex iterations	883	2065	2062	2800	4871
CPU [(a)]	1.029	2.854	3.385	4.336	8.127
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.232127	0.133975	0.066987	0.026795	0.013397
No. of non-zero R_i	20	56	56	56	56
Confidence level α	0.99	0.95	0.90	0.75	0.50
input data, Var.=4189, Cons.=4021, Nonz.=349555					
CVaR	-2.039670	-1.106810	-0.733006	-0.312627	-0.156314
VaR	-1.464350	-0.570700	-0.183289	0.000000	0.000000
Expected portfolio return	0.040122	0.029419	0.019309	0.014214	0.014214
No. of securities in portfolio	23	34	39	27	27
Dual simplex iterations	582	2217	3414	6058	8019
CPU [(a)]	0.702	2.730	3.993	9.048	12.854
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.575319	0.536106	0.549717	0.312627	0.156314
No. of non-zero R_i	30	184	384	557	557

[(a)] CPU seconds for proving optimality on a laptop with Intel Core 2 Duo T9300, 2.5GHz, RAM 4GB, CPLEX v.11.

Table 9.12: Nondominated solutions for the weighted-sum program **M1** for different confidence levels α and for 4020, 3500, 1000 historical input data

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.99$											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-2.1309	-2.0491	-2.0437	-2.0420	-2.0414	-2.0408	-2.0401	-2.0397	-2.0397	-2.0397
VaR	-23.6607	-1.6186	-1.5172	-1.5021	-1.4748	-1.4680	-1.4699	-1.4664	-1.4644	-1.4644	-1.4644
Expected return	0.2096	0.0578	0.0459	0.0443	0.0434	0.0430	0.0423	0.0410	0.0401	0.0401	0.0401
No. of assets	1	19	19	20	20	21	21	22	23	23	23
Dual simplex iter.	0	412	609	518	640	730	636	668	693	739	582
CPU	0.173	1.341	0.733	0.561	0.748	1.060	1.045	1.856	0.967	1.950	0.733
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5123	0.5320	0.5416	0.5672	0.5735	0.5709	0.5737	0.5753	0.5753	0.5753
No. of non-zero R_i	0	30	30	30	31	31	29	30	30	30	30
$\alpha = 0.95$											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-1.2846	-1.1268	-1.1134	-1.1091	-1.1084	-1.1075	-1.1070	-1.1068	-1.1068	-1.1068
VaR	-23.6607	-0.7719	-0.6147	-0.5743	-0.5708	-0.5712	-0.5660	-0.5611	-0.5677	-0.5692	-0.5707
Expected return	0.2096	0.0621	0.0390	0.0349	0.0326	0.0320	0.0310	0.0301	0.0295	0.0295	0.0294
No. of assets	1	32	39	37	34	35	34	35	35	35	34
Dual simplex iter.	0	1232	1426	1388	1745	1797	1784	1780	2148	2013	2217
CPU	0.234	0.780	3.276	0.951	1.872	1.872	1.669	1.840	2.917	2.995	4.118
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5127	0.5121	0.5391	0.5383	0.5372	0.5415	0.5459	0.5391	0.5376	0.5361
No. of non-zero R_i	0	183	182	184	185	185	186	182	185	186	180
$\alpha = 0.90$											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-1.0003	-0.7620	-0.7430	-0.7365	-0.7344	-0.7336	-0.7333	-0.7331	-0.7330	-0.7330
VaR	-23.6607	-0.4936	-0.2721	-0.2461	-0.2238	-0.2152	-0.2057	-0.1954	-0.1890	-0.1828	-0.1833
Expected return	0.2096	0.0683	0.0332	0.0273	0.0238	0.0221	0.0211	0.0206	0.0200	0.0195	0.0193
No. of assets	1	39	38	40	41	39	40	38	37	38	39
Dual simplex iter.	0	1912	2204	2639	2378	2480	2526	2926	3278	3349	3414
CPU	0.156	1.185	1.404	2.137	1.669	2.340	1.874	2.745	3.463	4.087	4.149
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.5068	0.4899	0.4969	0.5127	0.5193	0.5279	0.5379	0.5441	0.5502	0.5497
No. of non-zero R_i	0	388	384	379	383	380	383	383	383	384	391
$\alpha = 0.75$											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-0.8647	-0.3461	-0.3206	-0.3151	-0.3140	-0.3131	-0.3128	-0.3127	-0.3126	-0.3126
VaR	-23.6607	-0.2573	-0.0138	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Expected return	0.2096	0.1077	0.0282	0.0205	0.0178	0.0168	0.0157	0.0152	0.0150	0.0144	0.0142
No. of assets	1	34	52	25	26	27	27	27	27	27	27
Dual simplex iter.	0	3420	4191	4156	3904	3551	3961	4015	4885	6022	6058
CPU	0.171	1.326	2.090	2.496	2.683	2.012	3.416	2.854	5.023	11.856	9.235
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.6074	0.3323	0.3206	0.3151	0.3140	0.3131	0.3128	0.3127	0.3126	0.3126
No. of non-zero R_i	0	1089	723	554	596	549	453	489	553	478	557
$\alpha = 0.50$											
Var.=4189, Cons.=4021, Nonz.=349555											
CVaR	-23.6607	-0.7286	-0.3386	-0.1802	-0.1631	-0.1586	-0.1574	-0.1569	-0.1564	-0.1564	-0.1563
VaR	-23.6607	0.0454	0.0107	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Expected return	0.2096	0.1480	0.0794	0.0310	0.0226	0.0189	0.0176	0.0166	0.0152	0.0149	0.0142
No. of assets	1	20	41	47	25	26	26	27	27	27	27
Dual simplex iter.	0	4176	4715	5400	5031	4401	4310	4756	5611	6899	8019
CPU	0.171	1.185	1.622	2.636	2.418	1.934	2.277	3.042	4.820	9.703	13.104
$(1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i$	0.0000	0.7741	0.3493	0.1802	0.1631	0.1586	0.1574	0.1569	0.1564	0.1564	0.1563
No. of non-zero R_i	0	1729	1806	1112	546	346	547	547	553	553	557

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